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Asymptotics for Erdős–Solovej Zero Modes in Strong Fields

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Abstract. We consider the strong field asymptotics for the occurrence of zero modes of certain Weyl–Dirac operators on $\mathbb{R}^3$. In particular, we are interested in those operators $\mathcal{D}_B$ for which the associated magnetic field B is given by pulling back a two-form β from the sphere $\mathbb{S}^2$ to $\mathbb{R}^3$ using a combination of the Hopf fibration and inverse stereographic projection. If $\int_{\mathbb{S}^2} \beta \neq 0$, we show that

$$\sum_{0 \leq t \leq T} \dim \operatorname{Ker} \mathcal{D}_{tB} = \frac{T^2}{8\pi^2} \left| \int_{\mathbb{S}^2} \beta \right| \int_{\mathbb{S}^2} |\beta| + o(T^2)$$

as $T \to +\infty$. The result relies on Erdős and Solovej's characterisation of the spectrum of $\mathcal{D}_{tB}$ in terms of a family of Dirac operators on $\mathbb{S}^2$, together with information about the strong field localisation of the Aharonov–Casher zero modes of the latter.

1. Introduction

Suppose B is a (smooth) magnetic field on $\mathbb{R}^3$, viewed either as a divergence-free vector field $B = (B_1, B_2, B_3)$ or as a closed two-form

$$B = B_1 \,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + B_2 \,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + B_3 \,\mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

Choose a corresponding magnetic potential (or one-form) $A = A_1 \,\mathrm{d}x_1 + A_2 \,\mathrm{d}x_2 + A_3 \,\mathrm{d}x_3$ which generates B in the sense that $B = \mathrm{d}A$ (such potentials exist by Poincaré's Lemma). A Weyl–Dirac operator can then be defined by

$$\mathcal{D}_{\mathbb{R}^3,B} = \sum_{j=1}^{3} \sigma_j \,(-i\nabla_j - A_j), \tag{1}$$

where σ_1, σ_2 and σ_3 are the Pauli matrices and $\nabla = (\nabla_1, \nabla_2, \nabla_3)$ denotes the usual gradient operator on $\mathbb{R}^3$. The operator $\mathcal{D}_{\mathbb{R}^3,B}$ acts on two-component spinor fields, which, on $\mathbb{R}^3$, can be viewed simply as $\mathbb{C}^2$ valued functions. Standard arguments (see [19, Theorem 4.3] for example) show that $\mathcal{D}_{\mathbb{R}^3,B}$ is essentially self-adjoint on $\mathbb{C}_0^\infty$. We also use $\mathcal{D}_{\mathbb{R}^3,B}$ to denote the corresponding closure which is an unbounded self-adjoint operator on $L^2(\mathbb{R}^3,\mathbb{C}^2)$.

We are interested in the question of when 0 is an eigenvalue of $\mathcal{D}_{\mathbb{R}^3,B}$ or, equivalently, of determining when $\mathcal{D}_{\mathbb{R}^3,B}$ has a non-trivial kernel.

Definition. Any eigenfunction of $\mathcal{D}_{\mathbb{R}^3,B}$ corresponding to 0 is called a zero mode.

Remark. The potential A (and hence the operator $\mathcal{D}_{\mathbb{R}^3,B}$) is not uniquely determined by B. However if $\mathrm{d}A = B = \mathrm{d}A'$, then $A - A' = \mathrm{d}\phi$ for some $\phi \in C^\infty(\mathbb{R}^3)$ (using Poincaré's Lemma). Multiplication by $e^{i\phi}$ then establishes a unitary equivalence between the operators $\mathcal{D}_{\mathbb{R}^3,B}$ defined using the potentials A and A'. It follows that the spectral properties of $\mathcal{D}_{\mathbb{R}^3,B}$, and in particular the existence of zero modes, depend only on B.

Zero modes have been studied in a number of contexts in mathematical physics including the stability of matter [13,17] and chiral gauge theories [1, 2]. Most early works concentrated on the construction of explicit examples, including the original example [17], examples with arbitrary multiplicity [2], compact support [6] and a certain rotational type of symmetry ([10]; further details below). Some subsequent work moved towards studying the set of all zero mode producing fields (or potentials) within a given class; in particular, this set is nowhere dense [4,5] and is generically a sub-manifold of co-dimension one ([7]; slightly different classes of potentials were considered in these works).

To further our understanding of which fields produce zero modes, it is reasonable to consider the problem in various asymptotic regimes. We focus on the strong field regime (which, via a simple rescaling of the zero mode equation, is equivalent to the semi-classical regime). For a fixed field B define a counting function N_B by

$$\mathsf{N}_B(T) = \sum_{0\le t\le T} \dim \operatorname{Ker} \mathcal{D}_{\mathbb{R}^3,tB}$$

for any $T \in \mathbb{R}^+$. The behaviour of $\mathsf{N}_B(T)$ as $T \to +\infty$ is more regular than that of $\dim \operatorname{Ker} \mathcal{D}_{\mathbb{R}^3,tB}$ and clearly gives information about the occurrence of zero modes for strong fields.

In [9], an upper bound of the form $\mathsf{N}_B(T) \le C\|A\|_{L^3}^3 T^3$ was obtained, valid for any $T \ge 0$ and potential $A \in L^3$ (with $B = \mathrm{d}A$). The purpose of the present work is to determine the precise leading order asymptotic behaviour of $\mathsf{N}_B(T)$ as $T \to +\infty$ for a large class of symmetric magnetic fields first considered in [10]. Before defining this class, we need to introduce some supporting ideas and notations.

Let $\Omega^2(\mathbb{S}^2)$ denote the set of two-forms on $\mathbb{S}^2$ and let $\mathbf{v}_{\mathbb{S}^2} \in \Omega^2(\mathbb{S}^2)$ denote the standard volume two-form. Any $\beta \in \Omega^2(\mathbb{S}^2)$ can then be written as $\beta = f\mathbf{v}_{\mathbb{S}^2}$ for a unique $f \in C^\infty(\mathbb{S}^2)$. The flux of β is defined as

$$\Phi(\beta) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \beta = \frac{1}{2\pi}\int_{\mathbb{S}^2} f\mathbf{v}_{\mathbb{S}^2}.$$

We use $|\beta|$ to denote the (not necessarily smooth) two-form $|f|\mathbf{v}_{\mathbb{S}^2}$.

Definition. Let $h : \mathbb{S}^3 \to \mathbb{S}^2$ and $\pi : \mathbb{S}^3\backslash\{(0,0,0,-1)\} \to \mathbb{R}^3$ denote the Hopf fibration and stereographic projection, respectively. Set

$$\mathcal{B}'_{\mathrm{ES}} = \left\{(\pi^{-1})^* h^*\beta : \beta \in \Omega^2(\mathbb{S}^2),\ \Phi(\beta) \neq 0\right\},$$

(where * denotes pullback). Define $\mathcal{B}_{\mathrm{ES}}$ similarly except without the condition $\Phi(\beta) \neq 0$.

Elements of $\mathcal{B}_{\mathrm{ES}}$ are closed two-forms on $\mathbb{R}^3$ and can thus be viewed as magnetic fields (note that all two-forms on the $\mathbb{S}^2$ are closed). Furthermore, fields $B \in \mathcal{B}_{\mathrm{ES}}$ are smooth and satisfy bounds of the form $|B(x)| = O(|x|^{-4})$ as $|x| \to \infty$, while it is always possible to find a smooth potential A with $B = \mathrm{d}A$ which satisfies the bounds of the form $|A(x)| = O(|x|^{-3})$ as $|x| \to \infty$. It follows that fields in $\mathcal{B}_{\mathrm{ES}}$ (and their associated potentials) fall into the classes considered in [4,5,7].

Our main result is the following.

Theorem 1.1. *Let $B \in \mathcal{B}'_{ES}$ with $B = (\pi^{-1})^* h^*\beta$ for $\beta \in \Omega^2(\mathbb{S}^2)$. Then,*

$$\mathsf{N}_B(T) = \tfrac{1}{2}\,|\Phi(\beta)|\,\Phi(|\beta|)\,T^2 + o(T^2) \quad \text{as } T \to +\infty. \tag{2}$$

The lower asymptotic bound in (2), together with the explicit form of $\mathsf{N}_B(T)$ for the special case of the "constant" field $\beta = \mathbf{v}_{\mathbb{S}^2}$, was obtained in [18]. It is also clear where the argument for the upper bound in [9] may gain an order in T, although it remains unclear whether the $O(T^3)$ upper bound might yet be sharp for some magnetic field B.

Fields in $\mathcal{B}_{\mathrm{ES}}$ are invariant under the symmetry of $\mathbb{R}^3$ induced by the rotation of $\mathbb{S}^3$ along the $\mathbb{S}^1$ fibres of the Hopf fibration. The main work in [10] is to show how this symmetry can be used to express the spectrum of $\mathcal{D}_{\mathbb{R}^3,tB}$ in terms of the spectra of a family of Dirac operators on $\mathbb{S}^2$ (see Sect. 3 for further details). To calculate $\mathsf{N}_B(T)$, we need to consider eigenvalues of the latter with modulus up to $1/4$. Aharonov–Casher zero modes (see Theorem 2.1) correspond to an eigenvalue of 0 and contribute $\frac{1}{2}|\Phi(\beta)|^2$ to the leading order coefficient on the right hand side of (2); when β has a variable sign, the remaining part of this coefficient comes from "approximate zero modes" which arise from the localising effects of strong fields (see Sect. 4 for further details).

This paper is organised as follows. Some background on Dirac operators on $\mathbb{S}^2$ is outlined in Sect. 2, while the key results we require from [10] are stated at the start of Sect. 3. The proof of Theorem 1.1 is then reduced to determining the large k asymptotics of a spectral quantity $\mathsf{N}_B^{(k)}$ relating to a family of Dirac operators on $\mathbb{S}^2$; see (7) and Theorem 3.3.

The relatively straightforward lower bound in Theorem 3.3 is covered in Sect. 4. Necessary information about the asymptotic number of approximate zero modes for Dirac operators on $\mathbb{S}^2$ is given in Theorem 4.1 and justified in Sect. 8 using equivalent results for the plane (from [8]). Section 4 concludes with further estimates relating to approximate zero modes; some of the arguments rely on ideas from differential geometry and are deferred to Sect. 9.

The remaining sections are dedicated to the justification of the upper bound in Theorem 3.3. In Sect. 5, the quantity $\mathsf{N}_B^{(k)}$ is expressed as the number of eigenvalues of a (non-self-adjoint) operator $\mathbf{L}$ within a particular set; see Proposition 5.1. In turn, this is estimated from the singular values of $\mathbf{L}$ via Weyl's inequality; Sect. 6 is devoted to estimating the singular values while the argument is tied up in Sect. 7.

Notation. We use $\mathrm{spec}(T)$ to denote the set of eigenvalues of an operator T with entries repeated according to geometric multiplicity. The subset of positive eigenvalues is denoted by $\mathrm{spec}^+(T)$. General positive constants are denoted by C, with numerical subscripts used when we wish to keep track of specific constants in subsequent discussions. The open disc in $\mathbb{R}^2$ with radius r and centre 0 is denoted as $\mathbb{D}_r$, while I_2 denotes the 2×2 identity matrix.

2. Dirac Operators on $\mathbb{S}^2$

To discuss the Dirac operators on $\mathbb{S}^2$, we firstly recall some notions from Riemannian geometry as well as the idea of a spin^c structure (spin^c spinor bundles, Clifford multiplication and spin^c connections). A fuller introduction can be found in [12] (see also [10] for a discussion in a similar spirit to that presented here).

Let $\langle \cdot, \cdot \rangle_{\mathbb{S}^2}$ denote the standard Riemannian metric on (the tangent bundle of) $\mathbb{S}^2$, with corresponding norm $|\cdot|_{\mathbb{S}^2}$. The same symbols will be used for the induced metric on the exterior bundle $\wedge^* T^* \mathbb{S}^2$. For $n = 0, 1, 2$ let $\Omega^n(\mathbb{S}^2)$ denote the set of n-forms (that is, sections of the n-form bundle $\wedge^n T^* \mathbb{S}^2$). Note that, $\int_{\mathbb{S}^2} \mathbf{v}_{\mathbb{S}^2} = 4\pi$, while $|\beta| = |\beta|_{\mathbb{S}^2} \mathbf{v}_{\mathbb{S}^2}$ for any $\beta \in \Omega^2(\mathbb{S}^2)$.

A spin^c spinor bundle Ψ on $\mathbb{S}^2$ is a Hermitian vector bundle over $\mathbb{S}^2$ with fibre $\mathbb{C}^2$ on which we can define Clifford multiplication. The latter is a unitary map $\sigma : T^* \mathbb{S}^2 \to \mathrm{Hom}(\Psi)$ which satisfies

$$\sigma(\omega)\sigma(\rho) + \sigma(\rho)\sigma(\omega) = 2\langle \omega, \rho \rangle_{\mathbb{S}^2} I$$

for all 1-forms ω and ρ, where $\mathrm{Hom}(\Psi)$ denotes the set of endomorphisms on Ψ with the inner product given by $\langle A, B \rangle_{\mathrm{Hom}(\Psi)} = \frac{1}{2} \mathrm{tr}(A^* B)$, and $I \in \mathrm{Hom}(\Psi)$ is the identity. (Clifford multiplication gives a unitary representation of the Clifford algebra $\mathrm{Cl}(T_x^* \mathbb{S}^2)$ on $\mathbb{C}^2$ which is isomorphic to the standard representation and varies smoothly with $x \in \mathbb{S}^2$.) Clifford multiplication extends naturally as a linear isomorphism $\sigma : \wedge^* T^* \mathbb{S}^2 \to \mathrm{Hom}(\Psi)$; in particular,

$$\sigma(\omega)\sigma(\mathbf{v}_{\mathbb{S}^2}) + \sigma(\mathbf{v}_{\mathbb{S}^2})\sigma(\omega) = 0 \tag{3}$$

for any one-form ω, while $\sigma(\mathbf{v}_{\mathbb{S}^2})^2 = I$. The latter expression allows us to write $\Psi = L_+ \oplus L_-$ where the line bundles $L_\pm$ are defined by $\xi \in L_\pm$ iff $\sigma(\mathbf{v}_{\mathbb{S}^2})\xi = \pm\xi$. We use $\langle \cdot, \cdot \rangle_\Psi$ and $|\cdot|_\Psi$ to denote the (fibrewise) inner product and norm on Ψ, while $\Gamma(\Psi)$ is the space of spinors (sections of Ψ).

Associated to a spin^c spinor bundle Ψ is a line bundle which (for $\mathbb{S}^2$) is given as $L = \Psi \wedge \Psi$ (the determinant bundle of Ψ). This line bundle determines Ψ up to isomorphism (note that $H^2(\mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z}$ which has no two-torsion). On $\mathbb{S}^2$, there are infinitely many mutually non-isomorphic spin^c spinor bundles which we denote as $\Psi^{(k)}$ for $k \in \mathbb{Z}$, labelled so that the first Chern number of the associated line bundle satisfies $c_1(L^{(k)})[\mathbf{v}_{\mathbb{S}^2}] = 2k$.

Fix $k \in \mathbb{Z}$. A spin^c connection on $\Psi^{(k)}$ is a connection $\widetilde{\nabla}$ which is compatible with the Hermitian structure on $\Psi^{(k)}$ and the Clifford multiplication. For $\xi, \eta \in \Gamma(\Psi^{(k)})$ and $X \in T\mathbb{S}^2$, the former compatibility means

$$X\langle \xi, \eta\rangle_{\Psi^{(k)}} = \langle \widetilde{\nabla}_X \xi, \eta\rangle_{\Psi^{(k)}} + \langle \xi, \widetilde{\nabla}_X \eta\rangle_{\Psi^{(k)}},$$

while the latter means $[\widetilde{\nabla}_X, \sigma(\omega)] = \sigma(\nabla_X \omega)$ for all forms ω; here, ∇ is the Levi-Civita connection on $\mathbb{S}^2$ (for the metric $\langle \cdot, \cdot\rangle_{\mathbb{S}^2}$). As $\nabla_X \mathbf{v}_{\mathbb{S}^2} = 0$, we get

$$[\widetilde{\nabla}_X, \sigma(\mathbf{v}_{\mathbb{S}^2})] = 0. \tag{4}$$

A spin^c connection $\widetilde{\nabla}$ on $\Psi^{(k)}$ is uniquely determined by a choice of (Hermitian) connection on $L^{(k)}$. It follows that the set of all spin^c connections is an affine space modelled on $i\Omega^1(\mathbb{S}^2)$ (note that $L^{(k)}$ has structure group $U(1)$ with Lie algebra $i\mathbb{R}$). In particular, given $\widetilde{\nabla}$ any other spin^c connection on $\Psi^{(k)}$ can be written as $\widetilde{\nabla} - i\alpha$ for some $\alpha \in \Omega^1(\mathbb{S}^2)$.

The curvature of the connection $\widetilde{\nabla}$ can be viewed as the $\mathrm{Hom}(\Psi^{(k)})$ valued two-form given by

$$\widetilde{R}(X, Y)\xi = \widetilde{\nabla}_X \widetilde{\nabla}_Y \xi - \widetilde{\nabla}_Y \widetilde{\nabla}_X \xi - \widetilde{\nabla}_{[X,Y]}\xi$$

for all $X, Y \in T\mathbb{S}^2$ and $\xi \in \Psi^{(k)}$. The magnetic two-form of $\widetilde{\nabla}$ is then defined as $\beta = \frac{i}{2}\,\mathrm{Tr}(\widetilde{R}) \in \Omega^2(\mathbb{S}^2)$. The first Chern class of $L^{(k)}$ is the cohomology class of $\frac{1}{\pi}\beta$, so

$$\Phi(\beta) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \beta = \tfrac{1}{2}\, c_1(L^{(k)})[\mathbf{v}_{\mathbb{S}^2}] = k;$$

that is, the total flux of any magnetic two-form on $\Psi^{(k)}$ must be equal to k. This flux condition is also sufficient for a two-form to be the magnetic two-form of a spin^c connection on $\Psi^{(k)}$. More precisely, if $\beta' \in \Omega^2(\mathbb{S}^2)$ with $\Phi(\beta') = k$, then $\beta' = \beta + \mathrm{d}\alpha$ for some $\alpha \in \Omega^1(\mathbb{S}^2)$ (this follows from the Hodge decomposition theorem and the fact that the harmonic two-forms on $\mathbb{S}^2$ are simply the constant multiples of $\mathbf{v}_{\mathbb{S}^2}$). A straightforward calculation then shows that β' is the magnetic two-form associated with the spin^c connection $\widetilde{\nabla}' = \widetilde{\nabla} - i\alpha$. The choice of α is only unique up to the addition of a closed one-form.

The *Dirac operator* corresponding to a given spin^c connection $\widetilde{\nabla}$ on $\Psi^{(k)}$ is defined as $\mathcal{D} = -i\,\mathrm{Tr}\,\sigma\widetilde{\nabla}$. If $\{e_1, e_2\}$ is a local orthonormal frame (of vector fields) with corresponding dual frame $\{\theta_1, \theta_2\}$ (of one-forms), we can equivalently write

$$\mathcal{D} = -i\sigma(\theta_1)\widetilde{\nabla}_{e_1} - i\sigma(\theta_2)\widetilde{\nabla}_{e_2}.$$

The operator $\mathcal{D}$ maps $\Gamma(\Psi^{(k)}) \to \Gamma(\Psi^{(k)})$. Taking closures $\mathcal{D}$ becomes an (unbounded) self-adjoint operator on the L^2 sections of $\Psi^{(k)}$; we denote the latter by $\mathcal{H}$. Since $\mathcal{D}$ is a first-order elliptic differential operator on a compact manifold, it has a compact resolvent and discrete spectrum. Furthermore, (3) and (4) give

$$\mathcal{D}(\sigma(\mathbf{v}_{\mathbb{S}^2})\,\cdot\,) = -\sigma(\mathbf{v}_{\mathbb{S}^2})\mathcal{D}, \tag{5}$$

so the spectrum of $\mathcal{D}$ is symmetric about 0. Combined with the Aharonov–Casher theorem ([3]; see [10] for the $\mathbb{S}^2$ version), we then have the following.

Theorem 2.1. *For any Dirac operator $\mathcal{D}$ on $\Psi^{(k)}$, we have* $\dim \operatorname{Ker} \mathcal{D} = |k|$, *while the spectrum of $\mathcal{D}$ is symmetric about* 0.

Remark. For the decomposition $\Psi^{(k)} = L_+^{(k)} \oplus L_-^{(k)}$ (induced by $\sigma(\mathbf{v}_{\mathbb{S}^2})$) (5) leads to

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$$

with $\mathcal{D}_\pm : \Gamma(L_\pm^{(k)}) \to \Gamma(L_\mp^{(k)})$. The Aharonov–Casher theorem can then be viewed as a combination of the Atiyah–Singer index theorem and a vanishing theorem for $\mathcal{D}$; the former gives

$$\dim \operatorname{Ker} \mathcal{D}_+ - \dim \operatorname{Ker} \mathcal{D}_- = \tfrac{1}{2} c_1(L^{(k)})[\mathbf{v}_{\mathbb{S}^2}] = k,$$

while the latter forces either $\operatorname{Ker} \mathcal{D}_+$ or $\operatorname{Ker} \mathcal{D}_-$ to be trivial.

A straightforward calculation shows that the Dirac operator associated with the spin^c connection $\widetilde{\nabla}' = \widetilde{\nabla} - i\alpha$ is $\mathcal{D}' = \mathcal{D} - \sigma(\alpha)$. Dirac operators also satisfy a simple gauge transformation rule; if $\psi \in C^\infty(\mathbb{S}^2) = \Omega^0(\mathbb{S}^2)$, then

$$e^{i\psi}\mathcal{D}(e^{-i\psi}\cdot) = \mathcal{D} - \sigma(\,\mathrm{d}\psi),$$

the Dirac operator corresponding to the spin^c connection $\widetilde{\nabla} - i\,\mathrm{d}\psi$. In particular, the Dirac operators corresponding to the spin^c connections $\widetilde{\nabla}$ and $\widetilde{\nabla} - i\,\mathrm{d}\psi$ are unitarily equivalent and hence have the same spectrum. It follows that the spectrum of a Dirac operator on $\mathbb{S}^2$ is determined entirely by the magnetic two-form of the corresponding spin^c connection (note that $H^1(\mathbb{S}^2) = 0$, so $\mathrm{d}\Omega^0(\mathbb{S}^2)$ is precisely the set of closed one-forms).

Let $\widetilde{\nabla}^{(k)}$ denote a spin^c connection on $\Psi^{(k)}$ corresponding to the "constant" magnetic two-form $\frac{k}{2}\mathbf{v}_{\mathbb{S}^2}$ and let $\mathcal{D}^{(k)}$ denote the corresponding Dirac operator. If $\beta \in \Omega^2(\mathbb{S}^2)$ is any other two-form with $\Phi(\beta) = \Phi(\frac{k}{2}\mathbf{v}_{\mathbb{S}^2}) = k$, we can find $\alpha \in \Omega^1(\mathbb{S}^2)$ with $\beta = \frac{k}{2}\mathbf{v}_{\mathbb{S}^2} + \mathrm{d}\alpha$ (as above). The spin^c connection $\widetilde{\nabla}^{(k)} - i\alpha$ then has a magnetic two-form β and a corresponding Dirac operator

$$\mathcal{D}_\alpha^{(k)} = \mathcal{D}^{(k)} - \sigma(\alpha). \tag{6}$$

This operator is uniquely determined by β up to gauge (and hence unitary) equivalence. We can view α as generating the "non-constant" part of β.

The situation for the Dirac operators on $\mathbb{S}^3$ is rather simpler. All spin^c bundles on $\mathbb{S}^3$ are isomorphic to the trivial bundle $\mathbb{S}^3 \times \mathbb{C}^2$, while any closed two-form $b \in \Omega^2(\mathbb{S}^3)$ gives rise to a self-adjoint Dirac operator $\mathcal{D}_{\mathbb{S}^3,b}$, which is unique up to unitary equivalence; see [10] for further details.

3. Reduction to $\mathbb{S}^2$

Let $\beta \in \Omega^2(\mathbb{S}^2)$ with $\Phi(\beta) = 1$. From the above discussion, we can write $\beta = \frac{1}{2}\mathbf{v}_{\mathbb{S}^2} + \mathrm{d}\alpha$ for some $\alpha \in \Omega^1(\mathbb{S}^2)$. Also set $b = h^*\beta$, the closed two-form on $\mathbb{S}^3$ obtained by pulling back β using the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$. For $t \in \mathbb{R}$, the magnetic field tb is invariant under rotations of $\mathbb{S}^3$ along the level sets of h. This symmetry is inherited by the Dirac operator $\mathcal{D}_{\mathbb{S}^3,tb}$, which allows the spectrum of $\mathcal{D}_{\mathbb{S}^3,tb}$ to be expressed in terms of the spectra of a family of Dirac operators on $\mathbb{S}^2$. The following is a restatement of [10, Theorem 8.1] (note that the metric $\frac{1}{4}\langle\cdot,\cdot\rangle_{\mathbb{S}^2}$ is used in [10], so eigenvalues of Dirac operators on $\mathbb{S}^2$ must include an extra factor of 2 here).

Theorem 3.1. *For any $t \in \mathbb{R}$, the spectrum of $\mathcal{D}_{\mathbb{S}^3,tb}$ is*

$$\bigcup_{k\in\mathbb{Z}} \Sigma_k \cup \big\{-\tfrac{1}{2} + \sqrt{4\lambda^2 + (k-t)^2},\, -\tfrac{1}{2} - \sqrt{4\lambda^2 + (k-t)^2} \;:\; \lambda \in \operatorname{spec}^+\big(\mathcal{D}^{(k)}_{t\alpha}\big)\big\},$$

where Σ_k contains the number $-\frac{1}{2} - \operatorname{sgn}(k)\,(k-t)$ counted with multiplicity $|k|$ (so, $\Sigma_0 = \emptyset$). The multiplicity of an eigenvalue of $\mathcal{D}_{\mathbb{S}^3,tb}$ is equal to the number of times it appears in the above list when the elements of Σ_k and $\operatorname{spec}^+(\mathcal{D}^{(k)}_{t\alpha})$ are counted with their relevant multiplicities.

Set $B = (\pi^{-1})^*b = (\pi^{-1})^*h^*\beta \in \mathcal{B}'_{\mathrm{ES}}$. From [10, Theorem 8.7], we have the following link between the Dirac operators $\mathcal{D}_{\mathbb{S}^3,tb}$ and $\mathcal{D}_{\mathbb{R}^3,tB}$.

Theorem 3.2. *For any $t \in \mathbb{R}$, we have* $\dim\operatorname{Ker}\mathcal{D}_{\mathbb{R}^3,tB} = \dim\operatorname{Ker}\mathcal{D}_{\mathbb{S}^3,tb}$.

Consider the disjoint partition of $\mathbb{R}$ given by the intervals

$$\widetilde{\tau}_k = \begin{cases} (k-\frac{1}{2}, k+\frac{1}{2}] & \text{if } k > 0, \\ [-\frac{1}{2}, \frac{1}{2}] & \text{if } k = 0, \\ [k-\frac{1}{2}, k+\frac{1}{2}) & \text{if } k < 0, \end{cases}$$

for $k \in \mathbb{Z}$. Also, let $\tau_k = (k-1/2, k+1/2)$ and $\overline{\tau}_k = [k-1/2, k+1/2]$ denote the interior and closure of $\widetilde{\tau}_k$, respectively. To identify the contribution to N_B coming from $t \in \tau_k$ and $t \in \widetilde{\tau}_k$, set

$$\mathsf{M}^{(k)}_B = \sum_{t\in\tau_k} \dim\operatorname{Ker}\mathcal{D}_{\mathbb{R}^3,tB} \quad\text{and}\quad \mathsf{N}^{(k)}_B = \sum_{t\in\widetilde{\tau}_k} \dim\operatorname{Ker}\mathcal{D}_{\mathbb{R}^3,tB}.$$

From Theorems 3.1 and 3.2, it is clear that $\operatorname{Ker}\mathcal{D}_{\mathbb{R}^3,tB}$ is non-trivial precisely when there exists $k \in \mathbb{Z}$ so that either $0 \in \Sigma_k$ or $4\lambda^2 + (k-t)^2 = 1/4$ for some $\lambda \in \operatorname{spec}^+(\mathcal{D}^{(k)}_{t\alpha})$, with corresponding agreement of multiplicities. In the latter case, we have $\lambda > 0$ which forces $(k-t)^2 < 1/4$ or $t \in \tau_k$. It follows that

$$\mathsf{M}^{(k)}_B = \#\big\{(t,\lambda) : \lambda \in \operatorname{spec}^+\big(\mathcal{D}^{(k)}_{t\alpha}\big) \text{ and } 4\lambda^2 + (k-t)^2 = \tfrac{1}{4}\big\}.$$

We also know that 0 is contained in the spectrum of $\mathcal{D}^{(k)}_{t\alpha}$ with multiplicity $|k|$ for any $t \in \mathbb{R}$ (see Theorem 2.1), while $0^2 + (k-t)^2 = 1/4$ has two solutions ($t = k \pm 1/2$). Furthermore, the spectrum of $\mathcal{D}^{(k)}_{t\alpha}$ is symmetric about 0. Combining these observations, we get

$$\#\{(t,\lambda) : \lambda \in \mathrm{spec}\big(\mathcal{D}^{(k)}_{t\alpha}\big) \text{ and } 4\lambda^2 + (k-t)^2 = \tfrac{1}{4}\} = 2\mathsf{M}^{(k)}_B + 2|k|.$$

On the other hand, $0 \in \Sigma_k$, with multiplicity $|k|$, iff $t \in \widetilde{\tau}_k \backslash \tau_k$. It follows that $\mathsf{N}^{(k)}_B - \mathsf{M}^{(k)}_B = |k|$ and so

$$\mathsf{N}^{(k)}_B = \frac{1}{2}\#\{(t,\lambda) : \lambda \in \mathrm{spec}\big(\mathcal{D}^{(k)}_{t\alpha}\big) \text{ and } 4\lambda^2 + (k-t)^2 = \tfrac{1}{4}\}. \tag{7}$$

Clearly in calculating the right hand side of (7), we need only consider $t \in \overline{\tau}_k$ and eigenvalues of $\mathcal{D}^{(k)}_{t\alpha}$ in $[-1/4, 1/4]$. In addition to the eigenvalue 0 with multiplicity $|k|$ (the Aharonov–Casher zero modes), there may be small non-zero eigenvalues (the approximate zero modes). The total number of these eigenvalues can be determined asymptotically in $|k|$ (see Theorem 4.1) which ultimately leads to the following.

Theorem 3.3. *We have* $\mathsf{N}^{(k)}_B = \Phi(|\beta|)\,|k| + o(|k|)$ *as* $|k| \to \infty$.

The lower bound for $\mathsf{N}^{(k)}_B$ contained in Theorem 3.3 was given in [18] and is included here for completeness (see Sect. 4). The justification of the upper bound for $\mathsf{N}^{(k)}_B$ appears in Sect. 7.

Given Theorem 3.3, the proof of our main result is now straightforward.

Proof of Theorem 1.1. We can extend the definition of $\mathsf{N}_B(T)$ to cover $T < 0$ by summing over $T \le t \le 0$ in this case. Together with the scaling properties of (2), it thus suffices to restrict to the case $\Phi(\beta) = 1$ and prove

$$\mathsf{N}_B(T) = \tfrac{1}{2}\,\Phi(|\beta|)\,T^2 + o(T^2) \quad \text{as } T \to \pm\infty. \tag{8}$$

Now, let $T > 0$ and pick $k_T \in \mathbb{Z}$ with $T \in \widetilde{\tau}_{k_T}$. Then, $\bigcup_{k=1}^{k_T-1} \widetilde{\tau}_k \subset [0,T] \subset \bigcup_{k=0}^{k_T} \widetilde{\tau}_k$, so

$$\sum_{k=1}^{k_T-1} \mathsf{N}^{(k)}_B \le \mathsf{N}_B(T) \le \sum_{k=0}^{k_T} \mathsf{N}^{(k)}_B.$$

Using Theorem 3.3 and the fact that $|k_T - T| \le 1/2$, we get

$$\sum_{k=0}^{k_T} \mathsf{N}^{(k)}_B = \sum_{k=0}^{k_T} [\Phi(|\beta|)\,k + o(k)] = \tfrac{1}{2}\,\Phi(|\beta|)\,k_T^2 + o(k_T^2) = \tfrac{1}{2}\,\Phi(|\beta|)\,T^2 + o(T^2)$$

as $T \to +\infty$. Since the removal of the first and last terms from the sum will not change this asymptotic behaviour, (8) for $T > 0$ now follows. A similar argument clearly deals with the case $T < 0$. $\square$

4. The Lower Bound

Throughout the next four sections, we consider a fixed $\beta \in \Omega^2(\mathbb{S}^2)$ with $\Phi(\beta) = 1$ and write $\beta = \frac{1}{2}\mathbf{v}_{\mathbb{S}^2} + \mathrm{d}\alpha$ for some $\alpha \in \Omega^1(\mathbb{S}^2)$. For each $k \in \mathbb{Z}$ and $\varepsilon, R > 0$, set

$$\mathsf{n}(\varepsilon) = \mathsf{n}_{k,\alpha}(\varepsilon) = \#\big\{\lambda \in \operatorname{spec}\big(\mathcal{D}^{(k)}_{k\alpha}\big) : |\lambda| \le \varepsilon\big\}$$

(counting according to multiplicity) and

$$\mathsf{n}(\varepsilon, R) = \mathsf{n}_{k,\alpha}(\varepsilon, R) = \begin{cases} \mathsf{n}(R) - \mathsf{n}(\varepsilon) & \text{if } R \ge \varepsilon, \\ 0 & \text{if } R < \varepsilon. \end{cases}$$

As $\mathcal{D}^{(k)}_{k\alpha}$ has $|k|$ zero modes (recall Theorem 2.1), we have $\mathsf{n}(\varepsilon) \ge |k| = |\Phi(k\beta)|$; a strict inequality (for suitable ε) reflects the presence of approximate zero modes. In general, there will be $O(|k|)$ approximate zero modes whenever β has variable sign; more precisely, we have the following.

Theorem 4.1. *Suppose $\varepsilon_k = Ce^{-c|k|^\rho}$ for some $C, c > 0$ and $0 < \rho < 1$, while $R_k = o(|k|^{1/2})$ as $|k| \to \infty$. Then,*

$$\liminf_{|k|\to\infty} \frac{1}{|k|}\,\mathsf{n}_{k,\alpha}(\varepsilon_k) \ge \Phi(|\beta|) \quad \textit{and} \quad \limsup_{|k|\to\infty} \frac{1}{|k|}\,\mathsf{n}_{k,\alpha}(R_k) \le \Phi(|\beta|).$$

Hence, $\mathsf{n}_{k,\alpha}(\varepsilon_k) = \Phi(|\beta|)\,|k| + o(|k|)$ and $\mathsf{n}_{k,\alpha}(\varepsilon_k, R_k) = o(|k|)$ as $|k| \to \infty$.

The proof of this result is given in Sect. 8 where it is reduced to a similar result for the Pauli operator on a disc in $\mathbb{R}^2$.

From (6), we get

$$\mathcal{D}^{(k)}_{t\alpha} = \mathcal{D}^{(k)} - t\sigma(\alpha). \tag{9}$$

It follows that $t \mapsto \mathcal{D}^{(k)}_{t\alpha}$ defines a self-adjoint holomorphic family of operators. Using standard perturbation theory (see [16]), we can then choose real-analytic functions μ_n for $n \in \mathbb{Z}$ so that the full set of eigenvalues of $\mathcal{D}^{(k)}_{t\alpha}$ (including multiplicities) is $\{\mu_n(t) : n \in \mathbb{Z}\}$ for any $t \in \mathbb{R}$. We can now rewrite (7) as

$$\mathsf{N}^{(k)}_B = \frac{1}{2}\,\#\big\{(n,t) \in \mathbb{Z}\times\mathbb{R} \,:\, 4\mu_n^2(t) + (t-k)^2 = \tfrac{1}{4}\big\}. \tag{10}$$

Proof of the lower bound in Theorem 3.3. Fix $\varepsilon \in (0, 1/4)$ and suppose that $|\mu_n(k)| \le \varepsilon$ for some $n \in \mathbb{Z}$. Then, $4\mu_n^2(k) + (k-k)^2 \le 4\varepsilon^2 < 1/4$. However, μ_n is continuous and $4\mu_n^2(k \pm 1/2) + (k \pm 1/2 - k)^2 \ge 1/4$, so there are at least two values of t with $4\mu_n^2(t) + (t-k)^2 = 1/4$. From (10), it follows that

$$\mathsf{N}^{(k)}_B \ge \#\big\{n \in \mathbb{Z} : |\mu_n(k)| \le \varepsilon\big\} = \mathsf{n}_{k,\alpha}(\varepsilon). \tag{11}$$

The lower bound in Theorem 3.3 now follows from Theorem 4.1. $\square$

The complication with obtaining the upper bound in Theorem 3.3 is that, for each $n \in \mathbb{Z}$ with $\mu_n(k) < 1/4$, we need upper bounds on the number of values of t with $4\mu_n^2(t) + (t-k)^2 = 1/4$; in general, there is no reason why this cannot be more than two. We need some information about how rapidly $\mu_n(t)$ can change with respect to t.

Proposition 4.2. *For $j=1,2$ suppose λ_j is an eigenvalue of $\mathcal{D}^{(k)}_{t\alpha}$ with normalised eigenfunction ξ_j. Then, $|\langle \xi_1, \sigma(\alpha')\xi_2\rangle| \le \pi(|\lambda_1|+|\lambda_2|)\,\|\alpha'\|_{L^\infty}$ for any $\alpha' \in \Omega^1(\mathbb{S}^2)$.*

The proof of this result is given in Sect. 9.

Remark. If $\xi \in \operatorname{Ker} \mathcal{D}^{(k)}_{t\alpha}$, then Proposition 4.2 gives $\langle \xi, \sigma(\alpha')\xi\rangle = 0$ for any $\alpha' \in \Omega^1(\mathbb{S}^2)$, which forces the value of ξ to lie in either $L^{(k)}_+$ or $L^{(k)}_-$ at each point of $\mathbb{S}^2$. This result can be viewed as a local version of the vanishing theorem underlying the Aharonov–Casher theorem.

Corollary 4.3. *Set $a = 2\pi\|\alpha\|_{L^\infty}$. For any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, we have*

$$e^{-a|t-k|}|\mu_n(k)| \le |\mu_n(t)| \le e^{a|t-k|}|\mu_n(k)|.$$

Proof. Fix n. Since $\mathcal{D}^{(k)}_{t\alpha}$ is a self-adjoint holomorphic family, we can choose a normalised eigenfunction $\xi(t)$ for $\mu_n(t)$ which is real analytic in t (see [16]). Applying standard first-order perturbation theory to (9) then gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_n(t) = -\langle \xi(t), \sigma(\alpha)\xi(t)\rangle.$$

Thus, $|\mathrm{d}\mu_n/\mathrm{d}t| \le a|\mu_n|$ by Proposition 4.2. The result follows. □

Let $\varepsilon > 0$ and suppose $|\mu_n(k)| \le \varepsilon$ (ultimately we will use ε to control the size of the approximate zero modes of $\mathcal{D}^{(k)}_{k\alpha}$). For sufficiently small ε, Corollary 4.3 provides enough control over the behaviour of $\mu_n(t)$ when $t \in \overline{\tau}_k$ to ensure that there are precisely two values of t with $4\mu_n^2(t) + (t-k)^2 = 1/4$. Therefore, the issue of extra values of t can only arise when $\varepsilon < |\mu_n(k)| \le 1/4$. For reasonable choices of ε, Theorem 4.1 shows there are at most $o(|k|)$ such eigenvalues; we need to show that these eigenvalues lead to at most $o(|k|)$ extra values of t.

5. Linearisation

Our aim (Proposition 5.1) is to re-express the quantity $\mathsf{N}^{(k)}_B$ as the number of eigenvalues of some (compact non-self-adjoint) operator $\mathbf{L}$ within a prescribed set. In essence, this is achieved by using (9) and (10) to view $\mathsf{N}^{(k)}_B$ as the number of real eigenvalues of a quadratic spectral pencil and then linearising this pencil by moving to a suitably chosen 2×2 system.

Introduce a shifted parameter $s = t-k+1$. Then, $t \in \overline{\tau}_k$ iff $s \in J$, where $J = [1/2, 3/2]$. Also, set $\mathcal{D} = \mathcal{D}^{(k)}_{(k-1)\alpha}$ and $\mathcal{A} = \sigma(\alpha)$ so that (9) becomes

$$\mathcal{D}^{(k)}_{t\alpha} = \mathcal{D} - s\mathcal{A}.$$

Let $\mathbf{I} = I \otimes I_2$ denote the identity on $\mathcal{H}^2 = \mathcal{H} \otimes \mathbb{C}^2$. Introduce further operators $\mathbf{P}$ and $\mathbf{Q} = \mathbf{Q}_0 + \mathbf{Q}_1$ on $\mathcal{H}^2$ where

$$\mathbf{P} = 2\mathcal{D} \otimes \sigma_3 + I \otimes \sigma_1 - \tfrac{1}{2}\mathbf{I} = \begin{pmatrix} 2\mathcal{D} - \frac{1}{2}I & I \\ I & -2\mathcal{D} - \frac{1}{2}I \end{pmatrix},$$

$$\mathbf{Q}_0 = I \otimes \sigma_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q}_1 = 2\mathcal{A} \otimes \sigma_3 = \begin{pmatrix} 2\mathcal{A} & 0 \\ 0 & -2\mathcal{A} \end{pmatrix}.$$

In particular,

$$\mathbf{P} - s\mathbf{Q} = \begin{pmatrix} 2(\mathcal{D} - s\mathcal{A}) & (1-s)I \\ (1-s)I & -2(\mathcal{D} - s\mathcal{A}) \end{pmatrix} - \tfrac{1}{2}\mathbf{I}.$$

The operators $\mathbf{P}$ and $\mathbf{Q}$ are self-adjoint with $\mathbf{P}$ unbounded and $\mathbf{Q}$ bounded. In particular, $\operatorname{Dom}\mathbf{P} = (\operatorname{Dom}\mathcal{D})^2$, while $\mathbf{P} - s\mathbf{Q}$ has a compact resolvent for any $s \in \mathbb{R}$ (as $\mathcal{D} - s\mathcal{A}$ does). Also,

$$(\mathbf{P} - s\mathbf{Q} + \tfrac{1}{2}\mathbf{I})^2 = \left[4(\mathcal{D} - s\mathcal{A})^2 + (s-1)^2 I\right] \otimes I_2. \tag{12}$$

Taking $s = 0$, we get $(\mathbf{P} + \mathbf{I}/2)^2 \geq \mathbf{I}$ so that $|\mathbf{P}| \geq \mathbf{I}/2$, where $\mathbf{P} = |\mathbf{P}|\mathbf{U}$ is the polar decomposition of $\mathbf{P}$. It follows that $|\mathbf{P}|^{-1/2}$ is an injective compact operator with $\||\mathbf{P}|^{-1/2}\| \leq \sqrt{2}$. Define a further compact operator by

$$\mathbf{L} = \mathbf{U}|\mathbf{P}|^{-1/2}\mathbf{Q}|\mathbf{P}|^{-1/2}.$$

Let $C_1 = 4e^{a/2}$. For $0 \leq \varepsilon \leq 1/C_1$, set $s_\varepsilon^\pm = 1 \pm (1 - C_1\varepsilon)/2$; in particular, $J = [s_0^-, s_0^+]$. Also, set $J_\varepsilon^+ = [s_\varepsilon^+, s_0^+]$ and $J_\varepsilon^- = [s_0^-, s_\varepsilon^-]$.

Proposition 5.1. *We have*

$$\mathsf{N}_B^{(k)} = \frac{1}{2}\#\left\{\lambda \in \operatorname{spec}(\mathbf{L}) \,:\, \lambda^{-1} \in J\right\}. \tag{13}$$

Furthermore, if $0 < \varepsilon \leq 1/C_1$, *then*

$$\#\left\{\lambda \in \operatorname{spec}(\mathbf{L}) \,:\, \lambda^{-1} \in J_\varepsilon^\pm\right\} \geq \mathsf{n}(\varepsilon). \tag{14}$$

Approximate zero modes correspond to the eigenvalues of $\mathbf{L}$ with reciprocals in J_ε^+ and J_ε^-; (14) is the corresponding restatement of (11).

Proof. From (10) and (12), we get

$$\begin{aligned} \mathsf{N}_B^{(k)} &= \frac{1}{2}\sum_{s \in J} \#\left\{n \in \mathbb{Z} : 4\mu_n^2(s+k-1) + (s-1)^2 = \tfrac{1}{4}\right\} \\ &= \frac{1}{4}\sum_{s \in J} \dim\operatorname{Ker}\left[(\mathbf{P} - s\mathbf{Q} + \tfrac{1}{2}\mathbf{I})^2 - \tfrac{1}{4}\mathbf{I}\right]. \end{aligned}$$

Now, $(I \otimes \sigma_2)(\mathbf{P} - s\mathbf{Q})(I \otimes \sigma_2) = -(\mathbf{P} - s\mathbf{Q}) - \mathbf{I}$ (note that $\sigma_2^2 = I_2$, while $\sigma_2\sigma_j\sigma_2 = -\sigma_j$ for $j = 1, 3$). It follows that

$$\begin{aligned} \dim\operatorname{Ker}\left[(\mathbf{P} - s\mathbf{Q} + \tfrac{1}{2}\mathbf{I})^2 - \tfrac{1}{4}\mathbf{I}\right] &= \dim\operatorname{Ker}(\mathbf{P} - s\mathbf{Q}) + \dim\operatorname{Ker}(\mathbf{P} - s\mathbf{Q} + \mathbf{I}) \\ &= 2\dim\operatorname{Ker}(\mathbf{P} - s\mathbf{Q}). \end{aligned}$$

However, $\mathbf{I} - s\mathbf{L} = \mathbf{U}|\mathbf{P}|^{-1/2}(\mathbf{P} - s\mathbf{Q})|\mathbf{P}|^{-1/2}$ so $\dim\operatorname{Ker}(\mathbf{I} - s\mathbf{L}) = \dim\operatorname{Ker}(\mathbf{P} - s\mathbf{Q})$ for any s (recall that $|\mathbf{P}|^{-1/2}$ is injective). Combining the above gives (13).

Now, $|s_\varepsilon^\pm - 1| = (1-C_1\varepsilon)/2 \le 1/2$. If $|\mu_n(k)| \le \varepsilon \le 1/C_1$ for some $n \in \mathbb{Z}$, then

$$|\mu_n(k + s_\varepsilon^\pm - 1)| \le e^{a/2}|\mu_n(k)| \le \tfrac{1}{4}C_1\varepsilon$$

using Corollary 4.3. It follows that

$$4\mu_n^2(k + s_\varepsilon^\pm - 1) + (s_\varepsilon^\pm - 1)^2 \le \tfrac{1}{4}(C_1\varepsilon)^2 + \tfrac{1}{4}(1 - C_1\varepsilon)^2 \le \tfrac{1}{4}.$$

However, $4\mu_n^2(k + s_0^\pm - 1) + (s_0^\pm - 1)^2 \ge 1/4$ while μ_n is continuous. Thus, there is at least one $s \in J_\varepsilon^\pm$ with $4\mu_n^2(k + s - 1) + (s - 1)^2 = 1/4$. Since $\mathsf{n}(\varepsilon) = \#\{n \in \mathbb{Z} : |\mu_n(k)| \le \varepsilon\}$ estimate (14) now follows. $\square$

6. Estimates for Singular Values

Define compact self-adjoint operators by

$$\mathbf{K}_j = |\mathbf{P}|^{-1/2}\mathbf{Q}_j|\mathbf{P}|^{-1/2}, \quad j = 0, 1$$

and $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1$. Then, $\mathbf{L} = \mathbf{UK}$, so $\mathbf{L}^*\mathbf{L} = \mathbf{K}^2$; in particular, the singular values of $\mathbf{L}$ are simply the moduli of the eigenvalues of $\mathbf{K}$. To study the latter, we treat $\mathbf{K}_1$ as a perturbation of $\mathbf{K}_0$; in turn, the spectrum of $\mathbf{K}_0$ can be determined from that of $\mathcal{D}$.

For any $d \in \mathbb{R}$, let X_d denote the symmetric 2×2 matrix

$$X_d = \begin{pmatrix} 2d - \frac{1}{2} & 1 \\ 1 & -2d - \frac{1}{2} \end{pmatrix}.$$

The eigenvalues of X_d are $-1/2 + \Delta$ and $-1/2 - \Delta$, where $\Delta = \sqrt{4d^2 + 1} \ge 1$. Thus,

$$\||X_d|^{-1/2}\| = \left(\Delta - \tfrac{1}{2}\right)^{-1/2} \le \min\{\sqrt{2},\, |d|^{-1/2}\}. \tag{15}$$

Define a quadratic polynomial by

$$p_d(\lambda) = \lambda^2 + \Delta^{-1}\lambda + \tfrac{1}{4} - \Delta^2.$$

Then, $p_d(0) \le -3/4$ so p_d has one root of each sign; let $\kappa^\pm(d)$ denote the reciprocal of the root with sign ± 1. Note that $\kappa^+(0) = 2$ and $\kappa^-(0) = -2/3$.

Lemma 6.1. *The eigenvalues of the 2×2 matrix $|X_d|^{-1/2}\sigma_1|X_d|^{-1/2}$ are $\kappa^+(d)$ and $\kappa^-(d)$. Moreover, $\pm\kappa^\pm(d) \le \min\{2, |d|^{-1}\}$ and $|\kappa^\pm(d) - \kappa^\pm(0)| \le 16d^2$.*

Let $x_d^\pm \in \mathbb{C}^2$ denote a normalised eigenvector of $|X_d|^{-1/2}\sigma_1|X_d|^{-1/2}$ corresponding to $\kappa^\pm(d)$.

Proof. We have $\det(|X_d|\sigma_1) = -|\det X_d| = \frac{1}{4} - \Delta^2$, while $2\Delta|X_d| + X_d = (2\Delta^2 - \frac{1}{2})I_2$ so that $\mathrm{Tr}(2\Delta|X_d|\sigma_1) = -\mathrm{Tr}(X_d\sigma_1) = -2$. Thus, p_d is the characteristic polynomial of $|X_d|\sigma_1$ and hence $|X_d|^{1/2}\sigma_1|X_d|^{1/2}$. The first part of the result follows as $\sigma_1^{-1} = \sigma_1$, while the second part can then be obtained from (15) and the fact that $\|\sigma_1\| = 1$.

Let $\chi_d^\pm = 1/\kappa^\pm(d)$ denote the roots of p_d, in particular, $|\chi_d^\pm| \ge 1/2$. Now, $p_d(\lambda)$ is decreasing in d^2 for fixed $\lambda > 0$, so $\chi_d^+ \ge \chi_0^+ = 1/2$. Also,

$$p_d\left(\Delta^2 - \tfrac{1}{2}\right) \ge \Delta - \tfrac{1}{2}(1 + \Delta^{-1}) \ge 0$$

(recall that $\Delta \geq 1$), so $\chi_d^+ \leq \Delta^2 - 1/2$. Thus, $0 \leq \chi_d^+ - \chi_0^+ \leq \Delta^2 - 1 = 4d^2$. On the other hand, $\chi_d^+ + \chi_d^- = -\Delta^{-1}$ for any d, so

$$(\chi_d^+ - \chi_0^+) + (\chi_d^- - \chi_0^-) = 1 - \Delta^{-1} \in [0, 2d^2].$$

It follows that $|\chi_d^- - \chi_0^-| \leq 4d^2$. Combined, we then get

$$|\kappa^{\pm}(d) - \kappa^{\pm}(0)| = \frac{|\chi_d^{\pm} - \chi_0^{\pm}|}{|\chi_d^{\pm}|\,|\chi_0^{\pm}|} \leq \frac{4d^2}{(\frac{1}{2})(\frac{1}{2})} = 16d^2,$$

completing the result. □

Set $\nu_n = \mu_n(k-1)$ for $n \in \mathbb{Z}$ (the eigenvalues of $\mathcal{D}$). By Corollary 4.3, we have

$$e^{-a}|\mu_n(k)| \leq |\nu_n| \leq e^a|\mu_n(k)|. \tag{16}$$

Choose an orthonormal basis $\{\xi_n : n \in \mathbb{Z}\}$ of $\mathcal{H}$ with $\mathcal{D}\xi_n = \nu_n\xi_n$. For each $n \in \mathbb{Z}$, set $\kappa_n^{\pm} = \kappa^{\pm}(\nu_n)$ and $u_n^{\pm} = \xi_n \otimes x_{\nu_n}^{\pm} \in \mathcal{H}^2$. The definitions of $\mathbf{K}_0$ and X_d lead to $\mathbf{K}_0 u_n^{\pm} = \kappa_n^{\pm} u_n^{\pm}$, so, in particular, $\{u_n^+, u_n^- : n \in \mathbb{Z}\}$ is an eigenbasis for $\mathbf{K}_0$.

Set $\mathbb{M}_\varepsilon = \{n \in \mathbb{Z} : |\mu_n(k)| \leq \varepsilon\}$ and $\mathbb{M}'_R = \{n \in \mathbb{Z} : |\mu_n(k)| > R\}$ for any $\varepsilon, R > 0$. Let $\Pi_\varepsilon^{\pm}$, Π'_R and $\Pi_{\varepsilon,R}$ denote the (orthogonal) spectral projections of $\mathbf{K}_0$ with

$$\operatorname{Ran}\Pi_\varepsilon^{\pm} = \operatorname{Sp}\{u_n^{\pm} : n \in \mathbb{M}_\varepsilon\}, \quad \operatorname{Ran}\Pi'_R = \operatorname{Sp}\{u_n^+, u_n^- : n \in \mathbb{M}'_R\}$$

and $\Pi_{\varepsilon,R} = \mathbf{I} - \Pi_\varepsilon^+ - \Pi_\varepsilon^- - \Pi'_R$. Clearly,

$$\dim\operatorname{Ran}\Pi_\varepsilon^{\pm} = \#\mathbb{M}_\varepsilon = \mathsf{n}(\varepsilon) \quad \text{and} \quad \dim\operatorname{Ran}\Pi_{\varepsilon,R} = 2\mathsf{n}(\varepsilon, R).$$

Lemma 6.2. *We have $\pm\mathbf{K}_0\Pi_\varepsilon^{\pm} \geq 0$ for all $\varepsilon > 0$. Furthermore, there exist $C_{2,1}, C_{2,2} > 0$ so that $\big\|[\mathbf{K}_0 - \kappa^{\pm}(0)\mathbf{I}]\,\Pi_\varepsilon^{\pm}\big\| \leq C_{2,1}\varepsilon^2$ and $\|\mathbf{K}_0\Pi'_R\| \leq C_{2,2}R^{-1}$ for all $\varepsilon, R > 0$.*

Proof. We have $\pm\kappa_n^{\pm} > 0$ for all n, while Lemma 6.1 and (16) give

$$|\kappa_n^{\pm} - \kappa^{\pm}(0)| \leq 16\,\nu_n^2 \leq 16e^{2a}\varepsilon^2$$

for $n \in \mathbb{M}_\varepsilon$, and $|\kappa_n^{\pm}| \leq |\nu_n|^{-1} \leq e^a R^{-1}$ for $n \in \mathbb{M}'_R$. The result follows (with $C_{2,1} = 16e^{2a}$ and $C_{2,2} = e^a$). □

Next, we consider $\mathbf{K}_1$; we begin with estimates for $\mathbf{K}_1$ restricted to certain spectral subspaces of $\mathbf{K}_0$.

Lemma 6.3. *There exist $C_{3,1}$, $C_{3,2} > 0$ so that $\|\Pi_\varepsilon^{\pi_1}\mathbf{K}_1\Pi_\varepsilon^{\pi_2}\| \leq C_{3,1}\,\varepsilon\,\mathsf{n}(\varepsilon)$ and $\|\mathbf{K}_1\Pi'_R\| \leq C_{3,2}R^{-1/2}$ for all $\varepsilon, R > 0$ and $\pi_1, \pi_2 \in \{+,-\}$.*

Proof. Since $\{u_n^{\pm} : n \in \mathbb{M}_\varepsilon\}$ is an orthonormal basis for $\operatorname{Ran}\Pi_\varepsilon^{\pm}$, we have

$$\|\Pi_\varepsilon^{\pi_1}\mathbf{K}_1\Pi_\varepsilon^{\pi_2}\|^2 \leq \sum_{m,n\in\mathbb{M}_\varepsilon} |\langle u_m^{\pi_1}, \mathbf{K}_1 u_n^{\pi_2}\rangle|^2. \tag{17}$$

Now, the definitions of $\mathbf{K}_1$ and $\mathbf{Q}_1$ give

$$\begin{aligned}\langle u_m^{\pi_1}, \mathbf{K}_1 u_n^{\pi_2}\rangle &= \big\langle |\mathbf{P}|^{-1/2}u_m^{\pi_1}, \mathbf{Q}_1|\mathbf{P}|^{-1/2}u_n^{\pi_2}\big\rangle \\ &= 2\langle \xi_m, \mathcal{A}\xi_n\rangle\,\big\langle |X_{\nu_m}|^{-1/2}x_{\nu_m}^{\pi_1}, \sigma_3|X_{\nu_n}|^{-1/2}x_{\nu_n}^{\pi_2}\big\rangle.\end{aligned}$$

Note that $\|\sigma_3\| = 1$, so $|\langle |X_{\nu_m}|^{-1/2} x^{\pi_1}_{\nu_m}, \sigma_3 |X_{\nu_n}|^{-1/2} x^{\pi_2}_{\nu_n}\rangle| \le 2$ by (15). On the other hand, when $m, n \in \mathbb{M}_\varepsilon$, Proposition 4.2 and (16) give

$$|\langle \xi_m, \mathcal{A}\xi_n\rangle| = |\langle \xi_m, \sigma(\alpha)\xi_n\rangle| \le \pi(|\nu_m| + |\nu_n|)\,\|\alpha\|_{L^\infty} \le ae^a\varepsilon.$$

Therefore, $|\langle u^{\pi_1}_m, \mathbf{K}_1 u^{\pi_2}_n\rangle| \le C_{3,1}\varepsilon$ with $C_{3,1} = 4ae^a$. Since $\#\mathbb{M}_\varepsilon = \mathsf{n}(\varepsilon)$, the first part of the result now follows from (17).

Now, let $u \in \operatorname{Ran}\Pi'_R$. Then, $u = \sum_{n\in\mathbb{M}'_R} \xi_n \otimes z_n$ for some $z_n \in \mathbb{C}^2$, so

$$|\mathbf{P}|^{-1/2} u = \sum_{n\in\mathbb{M}'_R} \xi_n \otimes |X_{\nu_n}|^{-1/2} z_n.$$

For $n \in \mathbb{M}'_R$, (15) and (16) lead to

$$\||X_{\nu_n}|^{-1/2} z_n\|^2 \le |\nu_n|^{-1}\|z_n\|^2 \le e^a R^{-1}\|z_n\|^2.$$

Since $\{\xi_n : n \in \mathbb{M}'_R\}$ is an orthonormal set (in $\mathcal{H}$), it follows that

$$\||\mathbf{P}|^{-1/2} u\|^2 = \sum_{n\in\mathbb{M}'_R} \||X_{\nu_n}|^{-1/2} z_n\|^2 \le e^a R^{-1} \sum_{n\in\mathbb{M}'_R} \|z_n\|^2 = e^a R^{-1}\|u\|^2.$$

Therefore, $\||\mathbf{P}|^{-1/2}\Pi'_R\| \le e^{a/2}R^{-1/2}$. Recalling that $\||\mathbf{P}|^{-1/2}\| \le \sqrt{2}$ and $\|\mathbf{Q}_1\| = 2\|\mathcal{A}\| = 2\|\alpha\|_{L^\infty}$, the required estimate for $\|\mathbf{K}_1\Pi'_R\|$ now follows (with $C_{3,2} = 2\sqrt{2}e^{a/2}\|\alpha\|_{L^\infty}$). □

For $\varepsilon, R > 0$, set

$$\delta(\varepsilon, R) = C_{2,1}\varepsilon^2 + 4C_{3,1}\varepsilon\,\mathsf{n}(\varepsilon) + C_{2,2}R^{-1} + 2C_{3,2}R^{-1/2}. \tag{18}$$

Let $\{\lambda^+_n : n \in \mathbb{N}\}$ and $\{\lambda^-_n : n \in \mathbb{N}\}$ denote the sets of positive and negative eigenvalues of $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1$, enumerated to include multiplicities and ordered so that $\lambda^-_1 \le \lambda^-_2 \le \cdots < 0 < \cdots \le \lambda^+_2 \le \lambda^+_1$.

Proposition 6.4. *Suppose $\varepsilon, R > 0$. Then,*

$$\#\bigl\{n \in \mathbb{N} : |\lambda^\pm_n| > \pm\kappa^\pm(0) + \delta(\varepsilon, R)\bigr\} \le 2\mathsf{n}(\varepsilon, R) \tag{19}$$

and

$$\#\bigl\{n \in \mathbb{N} : |\lambda^\pm_n| > \delta(\varepsilon, R)\bigr\} \le \mathsf{n}(\varepsilon) + 2\mathsf{n}(\varepsilon, R). \tag{20}$$

The basic argument is a variational one with Lemmas 6.2 and 6.3 providing the relevant information about $\mathbf{K}_0$ and $\mathbf{K}_1$, respectively.

Proof. Set $M = \dim\operatorname{Ran}\Pi_{\varepsilon,R} = 2\mathsf{n}(\varepsilon, R)$. Let $H \le \mathcal{H}^2$ with $\dim H = M + 1$. Choose $u \in H$ with $\|u\| = 1$ and $\Pi_{\varepsilon,R}u = 0$. Then, $u = (\Pi^+_\varepsilon + \Pi^-_\varepsilon + \Pi'_R)u$, so

$$\langle u, \mathbf{K}_0 u\rangle = \langle u, \mathbf{K}_0\Pi^+_\varepsilon u\rangle + \langle u, \mathbf{K}_0\Pi^-_\varepsilon u\rangle + \langle u, \mathbf{K}_0\Pi'_R u\rangle \le \|\mathbf{K}_0\Pi^+_\varepsilon\| + \|\mathbf{K}_0\Pi'_R\|$$

(since $\mathbf{K}_0\Pi^-_\varepsilon \le 0$ from Lemma 6.2), while

$$\begin{aligned}\langle u, \mathbf{K}_1 u\rangle &= \langle u, (\Pi^+_\varepsilon + \Pi^-_\varepsilon)\mathbf{K}_1(\Pi^+_\varepsilon + \Pi^-_\varepsilon)u\rangle \\ &\quad + \langle u, \mathbf{K}_1\Pi'_R u\rangle + \langle \mathbf{K}_1\Pi'_R u, (\Pi^+_\varepsilon + \Pi^-_\varepsilon)u\rangle \\ &\le \|(\Pi^+_\varepsilon + \Pi^-_\varepsilon)\mathbf{K}_1(\Pi^+_\varepsilon + \Pi^-_\varepsilon)\| + 2\|\mathbf{K}_1\Pi'_R\|.\end{aligned}$$

Using a variational argument, it follows that

$$\lambda^+_{M+1} \le \|\mathbf{K}_0\Pi^+_\varepsilon\| + \|\mathbf{K}_0\Pi'_R\| + \|(\Pi^+_\varepsilon + \Pi^-_\varepsilon)\mathbf{K}_1(\Pi^+_\varepsilon + \Pi^-_\varepsilon)\| + 2\|\mathbf{K}_1\Pi'_R\|.$$

Lemmas 6.2 and 6.3 then give $\lambda^+_{M+1} \le \kappa^+(0) + \delta(\varepsilon, R)$. The case of the upper sign in (19) clearly follows. The lower sign can be obtained by a similar argument.

Now, set $M = \dim \operatorname{Ran}(\Pi^+_\varepsilon + \Pi_{\varepsilon,R}) = \mathsf{n}(\varepsilon) + 2\mathsf{n}(\varepsilon, R)$. A slightly simpler version of the above argument leads to

$$\begin{aligned}\lambda^+_{M+1} &\le \|\mathbf{K}_0\Pi'_R\| + \|\Pi^-_\varepsilon \mathbf{K}_1 \Pi^-_\varepsilon\| + 2\|\mathbf{K}_1\Pi'_R\| \\ &\le C_{2,2}R^{-1} + C_{3,1}\varepsilon\,\mathsf{n}(\varepsilon) + 2C_{3,2}R^{-1/2}.\end{aligned}$$

Since the final line is clearly bounded above by $\delta(\varepsilon, R)$ (20) now follows. □

7. The Upper Bound

The upper bound in Theorem 3.3 follows from Theorem 4.1 if we can show that $\mathsf{N}^{(k)}_B - \mathsf{n}_{k,\alpha}(\varepsilon_k)$ is bounded from above by $o(|k|)$ for suitably chosen ε_k. We first estimate this difference using Propositions 5.1 and 6.4 together with Weyl's inequality.

Lemma 7.1. *Suppose* $0 < \varepsilon \le 1/C_1$ *and* $R > 0$. *Then,*

$$\left[\tfrac{2}{3} - \delta(\varepsilon, R)\right]\left[\mathsf{N}^{(k)}_B - \mathsf{n}_{k,\alpha}(\varepsilon)\right] \le [\delta(\varepsilon, R) + C_1\varepsilon]\,\mathsf{n}_{k,\alpha}(\varepsilon) + 2\|\mathbf{K}\|\,\mathsf{n}_{k,\alpha}(\varepsilon, R).$$

Proof. Take $N = 2\mathsf{N}^{(k)}_B$ and $M = N - 2\mathsf{n}(\varepsilon)$. Let $\Lambda = \{\lambda \in \operatorname{spec}(\mathbf{L}) : \lambda^{-1} \in J\}$, so $\#\Lambda = N$ by Proposition 5.1. Also, let $\mathrm{K} \subset \operatorname{spec}(\mathbf{K})$ denote the collection of the N eigenvalues of $\mathbf{K}$ with largest moduli. Since $\mathbf{L}^*\mathbf{L} = \mathbf{K}^2$, the singular values of $\mathbf{L}$ are precisely the moduli of the eigenvalues of $\mathbf{K}$. Weyl's inequality ([20]) then gives

$$\sum_{\lambda\in\Lambda} \lambda \le \sum_{\lambda\in\mathrm{K}} |\lambda|. \tag{21}$$

For any $\lambda \in \Lambda$, we have $\lambda^{-1} \in J = [1/2, 3/2]$ so that $\lambda \ge 2/3$. If $\lambda^{-1} \in J^-_\varepsilon$, then

$$\lambda \ge \frac{1}{s^-_\varepsilon} = \frac{2}{1 + C_1\varepsilon} \ge 2\,(1 - C_1\varepsilon).$$

From Proposition 5.1, it follows that

$$\sum_{\lambda\in\Lambda} \lambda \ge 2\,(1 - C_1\varepsilon)\,\mathsf{n}(\varepsilon) + \tfrac{2}{3}[N - \mathsf{n}(\varepsilon)] = 2\big(\tfrac{4}{3} - C_1\varepsilon\big)\mathsf{n}(\varepsilon) + \tfrac{2}{3}M. \tag{22}$$

Write $\delta = \delta(\varepsilon, R)$. Proposition 6.4 shows that $\mathbf{K}$ has at most $2\mathsf{n}(\varepsilon, R)$ eigenvalues in each of the intervals $(-\infty, -2/3 - \delta)$ and $(2 + \delta, \infty)$, and at most $\mathsf{n}(\varepsilon) + 2\mathsf{n}(\varepsilon, R)$ eigenvalues in each of the intervals $(-\infty, -\delta)$ and (δ, ∞). Furthermore, the spectral radius of $\mathbf{K}$ is $\|\mathbf{K}\|$, while $\#\mathrm{K} - (2\mathsf{n}(\varepsilon) + 4\mathsf{n}(\varepsilon, R)) = M - 4\mathsf{n}(\varepsilon, R) \le M$. Therefore,

$$\sum_{\lambda\in\mathrm{K}} |\lambda| \le 4\|\mathbf{K}\|\,\mathsf{n}(\varepsilon, R) + \big(\tfrac{2}{3} + \delta\big)\mathsf{n}(\varepsilon) + (2 + \delta)\,\mathsf{n}(\varepsilon) + \delta M. \tag{23}$$

The result now follows when we combine (21), (22) and (23). □

Remark. The key to our argument is the identification of those eigenvalues and singular values of $\mathbf{L}$ which arise from the Aharonov–Casher and approximate zero modes. These contribute $\frac{8}{3}\mathsf{n}(\varepsilon)$ to each side of (21), the cancellation of which allows the quantity $\mathsf{N}_B^{(k)} - \mathsf{n}_{k,\alpha}(\varepsilon)$ to be estimated with sufficient precision.

Since $\||\mathbf{P}|^{-1/2}\| \le \sqrt{2}$, straightforward bounds on $\mathbf{Q}$ give

$$\|\mathbf{K}\| \le 2\|\mathbf{Q}_0 + \mathbf{Q}_1\| = 2(1 + 4\|\alpha\|_{L^\infty}^2)^{1/2}. \tag{24}$$

Proof of the upper bound in Theorem 3.3. Set $\varepsilon_k = e^{-|k|^{1/2}}$ and $R_k = |k|^{1/4}$ for all $k \in \mathbb{Z}$. As $|k| \to \infty$, we clearly have $\varepsilon_k = o(|k|^{-1})$ and $R_k \to \infty$, while Theorem 4.1 gives $\mathsf{n}(\varepsilon_k) = \Phi(|\beta|)|k| + o(|k|)$ and $\mathsf{n}(\varepsilon_k, R_k) = o(|k|)$. It follows that $\delta(\varepsilon_k, R_k) = o(1)$ (recall (18)) and so $\mathsf{N}_B^{(k)} - \mathsf{n}_{k,\alpha}(\varepsilon_k) = o(|k|)$ by Lemma 7.1 and (24). □

8. Approximate Zero Modes on $\mathbb{S}^2$

Let $\mathbb{S}^2_+$ (respectively, $\mathbb{S}^2_-$) denote the sphere with the south (respectively, north) pole removed; if we view $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$, then $\mathbb{S}^2_\pm = \mathbb{S}^2 \backslash \{(0,0,\mp 1)\}$. Let $z_\pm : \mathbb{S}^2_\pm \to \mathbb{R}^2$ denote the stereographic projection, given by

$$z_\pm(x) = \frac{1}{1 \pm x_3}(x_1, x_2), \quad x = (x_1, x_2, x_3) \in \mathbb{S}^2_\pm.$$

Set $\widetilde{\Omega}(x) = 2(1 + |x|^2)^{-1}$ for $x \in \mathbb{R}^2$, and $\Omega_\pm = \widetilde{\Omega} \circ z_\pm$. It is straightforward to check that the map $z_\pm$ is an isometry if $\mathbb{R}^2$ is given the conformal metric $\widetilde{\Omega}\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ (where $\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ is the usual Euclidean metric on $\mathbb{R}^2$). Hence, $z_\pm^*(\widetilde{\Omega}^2 \mathbf{v}_{\mathbb{R}^2}) = \mathbf{v}_{\mathbb{S}^2}$ (where $\mathbf{v}_{\mathbb{R}^2} = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ is the usual volume form on $\mathbb{R}^2$).

For any $\delta \in [0,1]$, set $\mathbb{S}^2_{\delta,\pm} = \mathbb{S}^2 \cap \{\pm x_3 < \delta\}$; in particular, $\mathbb{S}^2_{1,\pm} = \mathbb{S}^2_\pm$, while $\mathbb{S}^2_{0,+}$ and $\mathbb{S}^2_{0,-}$ are the north and south hemispheres. It is easy to check that $z_\pm(\mathbb{S}^2_{\delta,\pm}) = \mathbb{D}_{r_\delta}$ where $r_\delta^2 = (1+\delta)/(1-\delta)$, while we have the bounds

$$1 - \delta < \widetilde{\Omega}(x) \le 2, \quad x \in \mathbb{D}_{r_\delta}. \tag{25}$$

Using the isometry $z_\pm^{-1}$, we can pull back the (restricted) spin^c bundle $\Psi^{(k)}$ from $\mathbb{S}^2_\pm$ to get a spin^c bundle on $\mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, the latter is isomorphic to the trivial bundle $\mathbb{R}^2 \times \mathbb{C}^2$, so sections of this bundle (spinors) can be identified with maps $\mathbb{R}^2 \to \mathbb{C}^2$. For $\xi \in \Gamma(\Psi^{(k)})$ with $\mathrm{supp}(\xi) \subset \mathbb{S}^2_\pm$, let $\eta = \xi \circ z_\pm^{-1}$ denote the corresponding map in $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$. Then,

$$\|\xi\|^2_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2_\pm} |\xi|^2_{\Psi^{(k)}} \mathbf{v}_{\mathbb{S}^2} = \int_{\mathbb{R}^2} |\xi \circ z_\pm|^2 \,\widetilde{\Omega}^2 \mathbf{v}_{\mathbb{R}^2} = \|\widetilde{\Omega}\eta\|^2_{L^2(\mathbb{R}^2)}. \tag{26}$$

Using the isometry $z_\pm$ and the above identification of spin^c bundles, any Dirac operator on $\mathbb{S}^2$ can be restricted to $\mathbb{S}^2_\pm$ and then considered as a Dirac operator on $\mathbb{R}^2$ with the conformal metric $\widetilde{\Omega}\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$. Conformal mapping properties of Dirac operators (see [15, Section 1.4] or [10, Theorem 4.3]) mean that the latter is simply related to a Dirac operator on $\mathbb{R}^2$ with the usual metric. Under the above identification of spin^c bundles, a Dirac operator on $\mathbb{R}^2$ becomes a Weyl–Dirac operator corresponding to a potential $A' = A'_1\,\mathrm{d}x_1 + A'_2\,\mathrm{d}x_2$ on $\mathbb{R}^2$; that is, an operator given by the two-dimensional version of (1). More precisely, let $\alpha \in \Omega^1(\mathbb{S}^2)$ and consider the Dirac operator $\mathcal{D}_\alpha^{(k)}$ on $\Psi^{(k)}$. Then we can find $A_\pm \in \Omega^1(\mathbb{R}^2)$, so that

$$\big(\Omega_\pm^{3/2}\mathcal{D}_\alpha^{(k)}\Omega_\pm^{-1/2}\big)(\eta \circ z_\pm) = (\mathcal{D}_{\mathbb{R}^2,A_\pm}\eta) \circ z_\pm \tag{27}$$

for all $\eta : \mathbb{R}^2 \to \mathbb{C}^2$ (note that $\eta \circ z_\pm \in \Gamma(\Psi_\pm^{(k)})$, where $\Psi_\pm^{(k)}$ is the restriction of $\Psi^{(k)}$ to $\mathbb{S}^2_\pm$). Moreover, the magnetic field corresponding to $\mathcal{D}_{\mathbb{R}^2,A_\pm}$ is simply the pull-back of that corresponding to $\mathcal{D}_\alpha^{(k)}$ under the map $z_\pm^{-1}$; if the latter is $\beta = f\mathbf{v}_{\mathbb{S}^2}$, then the former will be given by $\beta_\pm = \mathrm{d}A_\pm = (f \circ z_\pm^{-1})\,\widetilde{\Omega}^2\,\mathbf{v}_{\mathbb{R}^2}$. In particular, for any open subset $U \subseteq \mathbb{R}^2$, we have

$$\int_U \beta_\pm = \int_{z_\pm^{-1}(U)} \beta. \tag{28}$$

For $A' \in \Omega^1(\mathbb{R}^2)$ and $r > 0$, let $\mathcal{P}_{\mathbb{D}_r,A'}$ denote the Pauli operator on $\mathbb{D}_r$ with magnetic potential A' and Dirichlet boundary conditions; this can be defined as the non-negative self-adjoint operator associated with the closure of the quadratic form given by $\eta \mapsto \|\mathcal{D}_{\mathbb{R}^2,A'}\eta\|^2_{L^2(\mathbb{R}^2)}$ for $\eta \in C_0^\infty(\mathbb{D}_r,\mathbb{C}^2)$.

For the next result, let $\mathcal{D}_\alpha^{(k)}$ denote a Dirac operator on $\Psi^{(k)}$ and let $A_\pm$ denote the corresponding one-forms on $\mathbb{R}^2$ as discussed above.

Proposition 8.1. *There exists $C_4 > 0$, so that, for any $\mu > 0$ and $\delta \in (0,1]$,*

$$\begin{aligned}&\#\{\lambda \in \operatorname{spec}(\mathcal{D}_\alpha^{(k)}) : |\lambda| \le \mu\}\\ &\qquad \ge \#\big\{\lambda \in \operatorname{spec}(\mathcal{P}_{\mathbb{D},A_+}) : \lambda \le \mu^2\big\} + \#\big\{\lambda \in \operatorname{spec}(\mathcal{P}_{\mathbb{D},A_-}) : \lambda \le \mu^2\big\}\end{aligned}$$

and

$$\begin{aligned}&\#\big\{\lambda \in \operatorname{spec}(\mathcal{D}_\alpha^{(k)}) : \lambda^2 \le \mu^2 - C_4\delta^{-2}\big\}\\ &\quad \le \#\big\{\lambda \in \operatorname{spec}(\mathcal{P}_{\mathbb{D}_{r_\delta},A_+}) : \lambda \le (4\mu)^2\big\} + \#\big\{\lambda \in \operatorname{spec}(\mathcal{P}_{\mathbb{D}_{r_\delta},A_-}) : \lambda \le (4\mu)^2\big\}.\end{aligned}$$

Proof. Let $\eta_\pm \in C_0^\infty(\mathbb{D},\mathbb{C}^2)$. Set $\xi_\pm = (\widetilde{\Omega}^{-1/2}\eta_\pm) \circ z_\pm^{-1}$ giving $\xi_\pm \in \Gamma(\Psi_\pm^{(k)})$ with $\operatorname{supp}(\xi_\pm) \subseteq \mathbb{S}^2_{0,\pm}$. Extend $\xi_\pm$ by 0 and set $\xi = \xi_+ + \xi_- \in \Gamma(\Psi^{(k)})$. From (25) we have $\widetilde{\Omega} \ge 1$ on $\mathbb{D}$. Together with (26) and (27), we then get

$$\|\xi_\pm\|^2_{L^2(\mathbb{S}^2_\pm)} = \big\|\widetilde{\Omega}^{1/2}\eta_\pm\big\|^2_{L^2(\mathbb{D})} \ge \|\eta_\pm\|^2_{L^2(\mathbb{D})}$$

and

$$\|\mathcal{D}_\alpha^{(k)}\xi_\pm\|^2_{L^2(\mathbb{S}^2_\pm)} = \big\|\widetilde{\Omega}^{-1/2}\mathcal{D}_{\mathbb{R}^2,A_\pm}\eta_\pm\big\|^2_{L^2(\mathbb{D})} \le \|\mathcal{D}_{\mathbb{D},A_\pm}\eta_\pm\|^2_{L^2(\mathbb{D})}.$$

Since ξ_+ and ξ_- have disjoint support, it follows that

$$\|\xi\|^2_{L^2(\mathbb{S}^2)} = \|\xi_+\|^2_{L^2(\mathbb{S}^2_+)} + \|\xi_-\|^2_{L^2(\mathbb{S}^2_-)} \ge \|\eta_+\|^2_{L^2(\mathbb{D})} + \|\eta_-\|^2_{L^2(\mathbb{D})}$$

and

$$\begin{aligned}\|\mathcal{D}^{(k)}_{\alpha}\xi\|^2_{L^2(\mathbb{S}^2)} &= \|\mathcal{D}^{(k)}_{\alpha}\xi_+\|^2_{L^2(\mathbb{S}^2_+)} + \|\mathcal{D}^{(k)}_{\alpha}\xi_-\|^2_{L^2(\mathbb{S}^2_-)}\\ &\le \|\mathcal{D}_{\mathbb{D},A_+}\eta_+\|^2_{L^2(\mathbb{D})} + \|\mathcal{D}_{\mathbb{D},A_-}\eta_-\|^2_{L^2(\mathbb{D})}.\end{aligned}$$

A standard variational argument then leads to the first part of the result.

Now choose non-negative $\chi_{\delta,\pm} \in C_0^\infty(\mathbb{S}^2_{\delta,\pm})$, so that $\chi^2_{\delta,+} + \chi^2_{\delta,-} = 1$ and $|d\chi_{\delta,\pm}| \le C_{4,0}\delta^{-1}$ on $\mathbb{S}^2$, where $C_{4,0}$ is independent of δ. Let $\xi \in \Gamma(\Psi^{(k)})$ and define compactly supported sections of $\Psi^{(k)}_\pm$ by setting $\xi_{\delta,\pm} = \chi_{\delta,\pm}\xi$. Also, set $\eta_{\delta,\pm} = \widetilde{\Omega}^{1/2}\xi_{\delta,\pm}\circ z_\pm$ giving $\eta_{\delta,\pm} \in C_0^\infty(\mathbb{D}_{r_\delta},\mathbb{C}^2)$. Then (26), (the upper bound in) (25) and (27) give

$$\|\xi_{\delta,\pm}\|^2_{L^2(\mathbb{S}^2_\pm)} = \big\|\widetilde{\Omega}^{1/2}\eta_{\delta,\pm}\big\|^2_{L^2(\mathbb{R}^2)} \le 2\|\eta_{\delta,\pm}\|^2_{L^2(\mathbb{D}_{r_\delta})},$$

so

$$\|\xi\|^2_{L^2(\mathbb{S}^2)} = \|\xi_{\delta,+}\|^2_{L^2(\mathbb{S}^2_+)} + \|\xi_{\delta,-}\|^2_{L^2(\mathbb{S}^2_-)} \le 2\big[\|\eta_{\delta,+}\|^2_{L^2(\mathbb{D}_{r_\delta})} + \|\eta_{\delta,-}\|^2_{L^2(\mathbb{D}_{r_\delta})}\big].$$

Similarly,

$$\|\mathcal{D}^{(k)}_{\alpha}\xi_{\delta,\pm}\|^2_{L^2(\mathbb{S}^2_\pm)} = \big\|\widetilde{\Omega}^{-1/2}\mathcal{D}_{\mathbb{R}^2,A_\pm}\eta_{\delta,\pm}\big\|^2_{L^2(\mathbb{R}^2)} \ge \tfrac{1}{2}\|\mathcal{D}_{\mathbb{D}_{r_\delta},A_\pm}\eta_{\delta,\pm}\|^2_{L^2(\mathbb{D}_{r_\delta})},$$

while

$$\begin{aligned}\|\mathcal{D}^{(k)}_{\alpha}\xi\|^2_{L^2(\mathbb{S}^2)} &= \|\chi_{\delta,+}\mathcal{D}^{(k)}_{\alpha}\xi\|^2_{L^2(\mathbb{S}^2)} + \|\chi_{\delta,-}\mathcal{D}^{(k)}_{\alpha}\xi\|^2_{L^2(\mathbb{S}^2)}\\ &= \big\|\mathcal{D}^{(k)}_{\alpha}\xi_{\delta,+} - i\sigma(d\chi_{\delta,+})\xi\big\|^2_{L^2(\mathbb{S}^2)} + \big\|\mathcal{D}^{(k)}_{\alpha}\xi_{\delta,-} - i\sigma(d\chi_{\delta,-})\xi\big\|^2_{L^2(\mathbb{S}^2)}\\ &\ge \tfrac{1}{2}\big[\|\mathcal{D}^{(k)}_{\alpha}\xi_{\delta,+}\|^2_{L^2(\mathbb{S}^2_+)} + \|\mathcal{D}^{(k)}_{\alpha}\xi_{\delta,-}\|^2_{L^2(\mathbb{S}^2_-)}\big] - 2C^2_{4,0}\delta^{-2}\|\xi\|^2_{L^2(\mathbb{S}^2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\mathcal{D}^{(k)}_{\alpha}\xi\|^2_{L^2(\mathbb{S}^2)} + 2C^2_{4,0}\delta^{-2}\|\xi\|^2_{L^2(\mathbb{S}^2)}\\ &\quad\ge \tfrac{1}{2}\big[\|\mathcal{D}_{\mathbb{D}_{r_\delta},A_+}\eta_{\delta,+}\|^2_{L^2(\mathbb{D}_{r_\delta})} + \|\mathcal{D}_{\mathbb{D}_{r_\delta},A_-}\eta_{\delta,-}\|^2_{L^2(\mathbb{D}_{r_\delta})}\big].\end{aligned}$$

A standard variational argument now completes the result (with $C_4 = 2C^2_{4,0}$; note that $\Gamma(\Psi^{(k)})$ is a core for $\mathcal{D}^{(k)}_{\alpha}$). □

We can use (27) to transfer results about approximate zero modes on $\mathbb{R}^2$ to $\mathbb{S}^2$; information about the former was obtained in [8].

Proof of Theorem 4.1. For each $k \in \mathbb{Z}$, we have a Dirac operator $\mathcal{D}^{(k)}_{k\alpha}$ on $\Psi^{(k)}$ with magnetic two-form $k\big(\frac{1}{2}\mathbf{v}_{\mathbb{S}^2} + \mathrm{d}\alpha\big)$. Pulling this back to $\mathbb{R}^2$ using $z_\pm$ as discussed above, we can arrange so that the corresponding one-forms on $\mathbb{R}^2$ are simply $kA_\pm$ for fixed (k independent) one-forms $A_\pm$. The corresponding field is $k\beta_\pm$ where $\beta_\pm = \mathrm{d}A_\pm$. By (28), we have

$$\int_{\mathbb{D}_{r_\delta}} |\beta_\pm| = \int_{\mathbb{S}^2_{\delta,\pm}} |\beta|, \qquad \delta \in [0,1]. \tag{29}$$

From [8, Theorem 1.2] and (29), we get

$$\liminf_{|k|\to\infty} \frac{1}{|k|}\,\#\big\{\lambda \in \mathrm{spec}(\mathcal{P}_{\mathbb{D},kA_\pm}) : \lambda \le \varepsilon_k^2\big\} \ge \frac{1}{2\pi}\int_{\mathbb{D}} |\beta_\pm| = \frac{1}{2\pi}\int_{\mathbb{S}^2_{0,\pm}} |\beta|.$$

Combined with Proposition 8.1, we then have

$$\liminf_{|k|\to\infty} \frac{1}{|k|}\, \mathsf{n}_{k,\alpha}(\varepsilon_k) \geq \frac{1}{2\pi}\int_{\mathbb{S}^2_{0,+}} |\beta| + \frac{1}{2\pi}\int_{\mathbb{S}^2_{0,-}} |\beta| = \Phi(|\beta|).$$

Now, let $\delta > 0$ and set $\widetilde{R}_k = 16(R_k^2 + C_4\delta^{-2})$ for $k \in \mathbb{Z}$. Then, $\widetilde{R}_k = o(|k|)$ as $|k| \to \infty$, so [8, Theorem 1.1] and (29) give

$$\limsup_{|k|\to\infty} \frac{1}{|k|}\, \#\{\lambda \in \operatorname{spec}(\mathcal{P}_{\mathbb{D}_{r_\delta},kA_\pm}) : \lambda \leq \widetilde{R}_k\} \leq \frac{1}{2\pi}\int_{\mathbb{D}_{r_\delta}} |\beta_\pm| = \frac{1}{2\pi}\int_{\mathbb{S}^2_{\delta,\pm}} |\beta|.$$

Combined with Proposition 8.1, we then have

$$\limsup_{|k|\to\infty} \frac{1}{|k|}\, \mathsf{n}_{k,\alpha}(R_k) \leq \frac{1}{2\pi}\int_{\mathbb{S}^2_{\delta,+}} |\beta| + \frac{1}{2\pi}\int_{\mathbb{S}^2_{\delta,-}} |\beta| = \Phi(|\beta|) + O(\delta)$$

as $\delta \to 0^+$ (note that β is bounded, while $|\mathbb{S}^2_{\delta,+} \cap \mathbb{S}^2_{\delta,-}| = O(\delta)$). Taking $\delta \to 0^+$ leads to the stated upper bound for $\mathsf{n}_{k,\alpha}(R_k)$. □

9. Spin-Field Estimates on $\mathbb{S}^2$

For any n, let $\mathrm{d} : \Omega^n(\mathbb{S}^2) \to \Omega^{n-1}(\mathbb{S}^2)$ and $\delta : \Omega^n(\mathbb{S}^2) \to \Omega^{n-1}(\mathbb{S}^2)$ denote the exterior derivative and its adjoint with respect to the Hodge $*$ operator. We have $* : \Omega^n(\mathbb{S}^2) \to \Omega^{2-n}(\mathbb{S}^2)$ with $** = (-1)^n$ and $\delta = -*\,\mathrm{d}\,*$. Also, $*\mathbf{v}_{\mathbb{S}^2} = 1$.

The expression $\mathrm{d}\delta + \delta\mathrm{d}$ defines the Laplace–de Rham operator on n-forms. For $n = 0$, this reduces to $\delta\mathrm{d} = -\Delta$, the negative of the Laplace–Beltrami operator on (scalar) functions. The Green's function for the latter is given in terms of $\log(1 - x.y)$ (where the dot product is defined by viewing $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$); more precisely, if $f \in C^\infty(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} f\mathbf{v}_{\mathbb{S}^2} = 0$, then

$$f(x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \log(1 - x.y)\, \Delta f(y)\, \mathbf{v}_{\mathbb{S}^2}(y) \tag{30}$$

for all $x \in \mathbb{S}^2$ (see [11, Theorem 4.15]). From this, we can obtain a related integral representation for one-forms. First, for any $y \in \mathbb{R}^3$, let $\rho_y \in \Omega^1(\mathbb{S}^2)$ denote the exterior derivative of $x \mapsto x.y$.

Proposition 9.1. *For any $\omega \in \Omega^1(\mathbb{S}^2)$ and $x \in \mathbb{S}^2$, we have*

$$\omega(x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{\rho_y(x)}{1 - x.y}\, \delta\omega(y)\, \mathbf{v}_{\mathbb{S}^2}(y) - \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{(*\rho_y)(x)}{1 - x.y}\, \mathrm{d}\omega(y).$$

Proof. Suppose $f \in C^\infty(\mathbb{S}^2)$ satisfies $\int_{\mathbb{S}^2} f\mathbf{v}_{\mathbb{S}^2} = 0$. Taking the exterior derivative of (30) with respect to x gives

$$\mathrm{d}f(x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{\rho_y(x)}{1 - x.y}\, \delta\mathrm{d}f(y)\, \mathbf{v}_{\mathbb{S}^2}(y). \tag{31}$$

Now, suppose $\nu \in \Omega^2(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} \nu = 0$. Set $g = *\nu \in C^\infty(\mathbb{S}^2)$, so that $\delta\nu = -*\,\mathrm{d}g$ and $(\delta\mathrm{d}g)\mathbf{v}_{\mathbb{S}^2} = \mathrm{d}\delta\nu$. Applying the Hodge $*$ to (31) then leads to

$$\delta\nu(x) = -\frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{(*\rho_y)(x)}{1 - x.y}\, \mathrm{d}\delta\nu(y). \tag{32}$$

Finally, suppose $\omega \in \Omega^1(\mathbb{S}^2)$. Since $H^1(\mathbb{S}^2) = 0$ the Hodge decomposition theorem gives $f \in C^\infty(\mathbb{S}^2)$ and $\nu \in \Omega^2(\mathbb{S}^2)$ such that $\omega = \mathrm{d}f + \delta\nu$. Since $\mathrm{d}1 = 0 = \delta\mathbf{v}_{\mathbb{S}^2}$, we may assume that $\int_{\mathbb{S}^2} f\,\mathbf{v}_{\mathbb{S}^2} = 0 = \int_{\mathbb{S}^2}\nu$. The result now follows from (31) and (32). □

For any $x, y \in \mathbb{S}^2$, it is easy to check $|\rho_y(x)|_{\mathbb{S}^2} = |(*\rho_y)(x)|_{\mathbb{S}^2} = 1-(x.y)^2$. A straightforward calculation then gives

$$\int_{\mathbb{S}^2}\left|\frac{\rho_y(x)}{1-x.y}\right|_{\mathbb{S}^2}\mathbf{v}_{\mathbb{S}^2}(x) = 2\pi^2 = \int_{\mathbb{S}^2}\left|\frac{(*\rho_y)(x)}{1-x.y}\right|_{\mathbb{S}^2}\mathbf{v}_{\mathbb{S}^2}(x).$$

Coupled with Proposition 9.1, we immediately get the following estimate for one-forms.

Corollary 9.2. *For any $\omega \in \Omega^1(\mathbb{S}^2)$, we have $\|\omega\|_{L^1} \le \frac{1}{2}\pi\,(\|\delta\omega\|_{L^1} + \|\,\mathrm{d}\omega\|_{L^1})$.*

When needed, $\{e_1, e_2\}$ denotes an orthonormal frame (of local vector fields), while $\{\theta_1, \theta_2\}$ denotes the corresponding orthonormal dual frame (of local one-forms). We assume $\{e_1, e_2\}$ is positively oriented so $\mathbf{v}_{\mathbb{S}^2} = \theta_1 \wedge \theta_2$. Also, $*\theta_1 = \theta_2$ and $*\theta_2 = -\theta_1$. For any $\omega \in \Omega^1(\mathbb{S}^2)$, we have the local expression

$$\delta\omega = -\operatorname{tr}\nabla\omega = -\big[(\nabla_{e_1}\omega)(e_1) + (\nabla_{e_2}\omega)(e_2)\big], \tag{33}$$

where ∇ denotes the Levi-Civita connection (on one-forms; see [14, Lemma 4.8]).

For any spinors $\xi, \eta \in \Gamma(\Psi^{(k)})$, let $\omega_{\xi,\eta} \in \Omega^1(\mathbb{S}^2)$ be the unique one-form satisfying

$$\langle\omega_{\xi,\eta}, \rho\rangle_{\mathbb{S}^2} = \langle\xi, \sigma(\rho)\eta\rangle_{\Psi^{(k)}}$$

for all $\rho \in \Omega^1(\mathbb{S}^2)$. In terms of a local orthonormal frame, we can write

$$\omega_{\xi,\eta} = \langle\xi, \sigma(\theta_1)\eta\rangle_{\Psi^{(k)}}\theta_1 + \langle\xi, \sigma(\theta_2)\eta\rangle_{\Psi^{(k)}}\theta_2.$$

Lemma 9.3. *Let $\widetilde{\nabla}$ be a spin^c connection on $\Psi^{(k)}$. If $\xi, \eta \in \Gamma(\Psi^{(k)})$ and $X \in \Gamma(T\mathbb{S}^2)$, then $\nabla_X\omega_{\xi,\eta} = \omega_{\widetilde{\nabla}_X\xi,\eta} + \omega_{\xi,\widetilde{\nabla}_X\eta}$.*

Proof. We have $X\langle\omega_{\xi,\eta}, \rho\rangle_{\mathbb{S}^2} = \langle\nabla_X\omega_{\xi,\eta}, \rho\rangle_{\mathbb{S}^2} + \langle\omega_{\xi,\eta}, \nabla_X\rho\rangle_{\mathbb{S}^2}$, while

$$\begin{aligned} X\langle\xi, \sigma(\rho)\eta\rangle_{\Psi^{(k)}} &= \langle\widetilde{\nabla}_X, \xi\sigma(\rho)\eta\rangle_{\Psi^{(k)}} + \langle\xi, \widetilde{\nabla}_X(\sigma(\rho)\eta)\rangle_{\Psi^{(k)}} \\ &= \langle\widetilde{\nabla}_X, \xi\sigma(\rho)\eta\rangle_{\Psi^{(k)}} + \langle\xi, \sigma(\rho)\widetilde{\nabla}_X\eta\rangle_{\Psi^{(k)}} + \langle\xi, \sigma(\nabla_X\rho)\eta\rangle_{\Psi^{(k)}}. \end{aligned}$$

The result now follows from the definition of $\omega_{\xi,\eta}$. □

Recall that Clifford multiplication extends naturally to two-forms; in particular, $\sigma(\mathbf{v}_{\mathbb{S}^2}) = \sigma(\theta_1)\sigma(\theta_2)$, while for any one-form ρ

$$\sigma(\rho)\sigma(\mathbf{v}_{\mathbb{S}^2}) = -\sigma(*\rho). \tag{34}$$

Proposition 9.4. *Let $\mathcal{D}$ be a Dirac operator on $\Psi^{(k)}$. If $\xi, \eta \in \Gamma(\Psi^{(k)})$, then*

$$\delta\omega_{\xi,\eta} = i\langle\mathcal{D}\xi, \eta\rangle_{\Psi^{(k)}} - i\langle\xi, \mathcal{D}\eta\rangle_{\Psi^{(k)}} \tag{35}$$

and

$$\mathrm{d}\omega_{\xi,\eta} = -i\big[\langle \mathcal{D}\xi, \sigma(\mathbf{v}_{\mathbb{S}^2})\eta\rangle_{\Psi^{(k)}} + \langle \xi, \sigma(\mathbf{v}_{\mathbb{S}^2})\mathcal{D}\eta\rangle_{\Psi^{(k)}}\big]\mathbf{v}_{\mathbb{S}^2}. \tag{36}$$

Proof. Let $\widetilde{\nabla}$ denote the spin^c connection defining $\mathcal{D}$. By (33) and Lemma 9.3,

$$\begin{aligned}
\delta\omega_{\xi,\eta} &= -\nabla_{e_1}\omega_{\xi,\eta}(e_1) - \nabla_{e_2}\omega_{\xi,\eta}(e_2)\\
&= -\omega_{\widetilde{\nabla}_{e_1}\xi,\eta}(e_1) - \omega_{\widetilde{\nabla}_{e_2}\xi,\eta}(e_2) - \omega_{\xi,\widetilde{\nabla}_{e_1}\eta}(e_1) - \omega_{\xi,\widetilde{\nabla}_{e_2}\eta}(e_2)\\
&= -\big\langle \big[\sigma(\theta_1)\widetilde{\nabla}_{e_1} + \sigma(\theta_2)\widetilde{\nabla}_{e_2}\big]\xi, \eta\big\rangle_{\Psi^{(k)}} - \big\langle \xi, \big[\sigma(\theta_1)\widetilde{\nabla}_{e_1} + \sigma(\theta_2)\widetilde{\nabla}_{e_2}\big]\eta\big\rangle_{\Psi^{(k)}}\\
&= -\langle i\mathcal{D}\xi, \eta\rangle_{\Psi^{(k)}} - \langle \xi, i\mathcal{D}\eta\rangle_{\Psi^{(k)}}.
\end{aligned}$$

On the other hand, working in a local orthonormal frame and applying (34) give

$$\begin{aligned}
\omega_{\xi,\eta} &= \langle \xi, \sigma(-\theta_2)\eta\rangle_{\Psi^{(k)}} *\theta_1 + \langle \xi, \sigma(*\theta_1)\eta\rangle_{\Psi^{(k)}} *\theta_2\\
&= -\langle \xi, \sigma(\theta_2)\sigma(\mathbf{v}_{\mathbb{S}^2})\eta\rangle_{\Psi^{(k)}}\theta_2 + \langle \xi, \sigma(\theta_1)\sigma(\mathbf{v}_{\mathbb{S}^2})\eta\rangle_{\Psi^{(k)}}(-\theta_1) = \omega_{\xi,\sigma(\mathbf{v}_{\mathbb{S}^2})\eta}.
\end{aligned}$$

Together with (5) and (35), we get

$$\delta*\omega_{\xi,\eta} = -\delta\omega_{\xi,\sigma(\mathbf{v}_{\mathbb{S}^2})\eta} = i\langle \mathcal{D}\xi, \sigma(\mathbf{v}_{\mathbb{S}^2})\eta\rangle_{\Psi^{(k)}} - i\langle \xi, -\sigma(\mathbf{v}_{\mathbb{S}^2})\mathcal{D}\eta\rangle_{\Psi^{(k)}}.$$

However, $\mathrm{d} = - * \delta *$ and $*1 = \mathbf{v}_{\mathbb{S}^2}$, so (36) follows. □

Proof of Proposition 4.2. Define a vector field X' on $\mathbb{S}^2$ by $\alpha' = \langle X', \cdot\rangle_{\mathbb{S}^2}$. Then, $|X'|_{\mathbb{S}^2} = |\alpha'|_{\mathbb{S}^2}$, while $\langle \xi_1, \sigma(\alpha')\xi_2\rangle_{\Psi^{(k)}} = \omega_{\xi_1,\xi_2}(X')$. Hence,

$$\begin{aligned}
|\langle \xi_1, \sigma(\alpha')\xi_2\rangle| &\le \int_{\mathbb{S}^2} |X'|_{\mathbb{S}^2}\, |\omega_{\xi_1,\xi_2}|_{\mathbb{S}^2}\, \mathbf{v}_{\mathbb{S}^2}\\
&\le \|\alpha'\|_{L^\infty(\mathbb{S}^2)}\, \|\omega_{\xi_1,\xi_2}\|_{L^1(\mathbb{S}^2)}\\
&\le \frac{\pi}{2}\, \|\alpha'\|_{L^\infty(\mathbb{S}^2)}\big[\|\delta\omega_{\xi_1,\xi_2}\|_{L^1(\mathbb{S}^2)} + \|\mathrm{d}\omega_{\xi_1,\xi_2}\|_{L^1(\mathbb{S}^2)}\big]
\end{aligned}$$

by Corollary 9.2. On the other hand, Proposition 9.4 leads to

$$\begin{aligned}
|\delta\omega_{\xi_1,\xi_2}|_{\mathbb{S}^2},\, |\mathrm{d}\omega_{\xi_1,\xi_2}|_{\mathbb{S}^2} &\le \big|\mathcal{D}^{(k)}_{t\alpha}\xi_1\big|_{\Psi^{(k)}}\, |\xi_2|_{\Psi^{(k)}} + |\xi_1|_{\Psi^{(k)}}\, \big|\mathcal{D}^{(k)}_{t\alpha}\xi_2\big|_{\Psi^{(k)}}\\
&= (|\lambda_1| + |\lambda_2|)\, |\xi_1|_{\Psi^{(k)}}\, |\xi_2|_{\Psi^{(k)}}
\end{aligned}$$

(note that $\sigma(\mathbf{v}_{\mathbb{S}^2})$ is a unitary operator in the fibres of $\Psi^{(k)}$). However,

$$\begin{aligned}
2\int_{\mathbb{S}^2} |\xi_1|_{\Psi^{(k)}}\, |\xi_2|_{\Psi^{(k)}}\, \mathbf{v}_{\mathbb{S}^2} &\le \int_{\mathbb{S}^2} \big[|\xi_1|^2_{\Psi^{(k)}} + |\xi_2|^2_{\Psi^{(k)}}\big]\, \mathbf{v}_{\mathbb{S}^2}\\
&= \|\xi_1\|^2_{L^2(\mathbb{S}^2)} + \|\xi_2\|^2_{L^2(\mathbb{S}^2)} = 2.
\end{aligned}$$

The result follows. □

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